Quantum Algorithms Tutorial

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Post-quantum cryptography

- Quantum computers can break public-key cryptography that is based on assuming hardness of factoring, discrete logs, and a few other problems
- Post-quantum cryptography tries to design classical crypto schemes that cannot be broken efficiently by quantum algorithms
- Classical codemakers vs quantum codebreakers
- This tutorial:

  Get to know your enemy!

Quantum bits

- Richard Feynman,
  David Deutsch
  in early 1980s:


  Harness quantum effects for useful computations!

- Classical bit is 0 or 1; quantum bit is superposition of 0 and 1

  For example, can use an electron as qubit,
  with 0 = "spin up" and 1 = "spin down"

- 2 qubits is superposition of 4 basis states (00,01,10,11)
  3 qubits is superposition of 8 basis states (000,001, . . .)
  . . .
  1000 qubits: superposition of 2^{1000} states

- Massive space for computation! Easier said than done. . .

A bit of math: states

- 1-qubit basis states: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Qubit: superposition $\alpha_0|0\rangle + \alpha_1|1\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \in \mathbb{C}^2$
- 2-qubit basis state: $|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
- n-qubit state: $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle \in \mathbb{C}^{2^n}$
- Axiom: measuring state $|\psi\rangle$ gives $|x\rangle$ with probability $|\alpha_x|^2$
- Hence $\sum_{x\in\{0,1\}^n} |\alpha_x|^2 = 1$, so $|\psi\rangle$ is a vector of length 1

A bit of math: operations

- Quantum operation maps quantum states to quantum states and is *linear* $\Longrightarrow$ corresponds to unitary matrix
- Example 1-qubit gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix}$$

- More quantum: Hadamard gate $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad H|1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

But $H\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}H|0\rangle + \frac{1}{\sqrt{2}}H|1\rangle = |0\rangle$

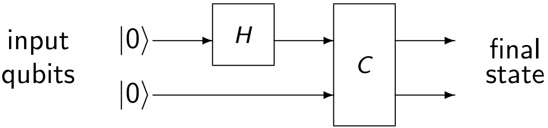
Interference!

- Controlled-NOT gate on 2 qubits: $|a, b\rangle \mapsto |a, a \oplus b\rangle$

Quantum circuits

- A classical Boolean circuit consists of AND, OR, and NOT gates on an n-bit register
- A quantum circuit consists of unitary quantum gates on an n-qubit register (allowing H, T, and CNOT gates suffices)

Example:



$$|00\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This circuit creates an EPR-pair: entanglement!

Recap: From classical to quantum computation

- bits $\longrightarrow$ qubits
- AND/OR/NOT gates $\longrightarrow$ unitary quantum gates
- classical circuit $\longrightarrow$ quantum circuit
- reading the output bit $\longrightarrow$ measuring final state

Quantum mechanical computers

1. Start with all qubits in easily-preparable state (e.g. all $|0\rangle$)
2. Run a circuit that produces the right kind of interference: computational paths leading to correct output should interfere constructively, others should interfere destructively
3. Measurement of final state gives classical output

Two important questions:

1. Can we build such a computer?
2. What can it do?

   This tutorial: 2nd question, focus on quantum algorithms

Quantum parallelism

- Suppose classical algorithm computes $f : \{0,1\}^n \to \{0,1\}^m$
- Convert this to quantum circuit $U : |x\rangle|0\rangle \mapsto |x\rangle|f(x)\rangle$
- We can now compute f "on all inputs simultaneously"!

$$U\left(\frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle|0\rangle\right) = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle|f(x)\rangle$$

- This contains all 2^n function values!
- But observing gives only one random $|x\rangle|f(x)\rangle$
  All other information will be lost
- More tricks needed for successful quantum computation

Interference!

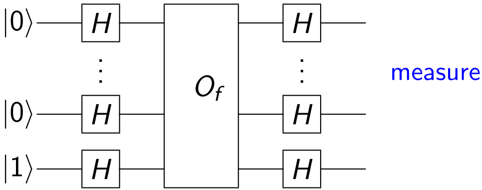
Deutsch-Jozsa problem

- Given: function $f : \{0,1\}^n \to \{0,1\}$ (2^n bits) , s.t.
  (1) $f(x) = 0$ for all x (constant),
  or
  (2) $f(x) = 0$ for $\frac{1}{2} \cdot 2^n$ of the x's (balanced)
- Question: is f constant or balanced?

- Classically: need at least $\frac{1}{2} \cdot 2^n + 1$ steps ("queries" to f)
- Quantumly: $O(n)$ gates suffice, and only 1 query

- Query: application of unitary $O_f : |x, 0\rangle \mapsto |x, f(x)\rangle$
- More generally: $O_f : |x, b\rangle \mapsto |x, b \oplus f(x)\rangle$ ($b \in \{0,1\}$)
- NB using $|-\rangle = H|1\rangle$, we can get queried bit as a $\pm$-phase:
  $O_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

Deutsch-Jozsa algorithm



- Starting state: $|\underbrace{0\ldots0}_{n}\rangle|1\rangle$
- After first Hadamards: $\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle|-\rangle$
- Make one query: $\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle|-\rangle$
- Forget about the last qubit $|-\rangle$

Deutsch-Jozsa (continued)

- After second Hadamard:

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

- $\alpha_{0\ldots0} = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} = \begin{cases} 1 & \text{if constant} \\ 0 & \text{if balanced} \end{cases}$
- Measurement gives right answer with certainty
- Big quantum-classical separation: $O(n)$ vs $\Omega(2^n)$ steps
- But the problem is efficiently solvable by bounded-error classical algorithm (just query f at a few random x)

The meat of this tutorial: 4 quantum algorithms

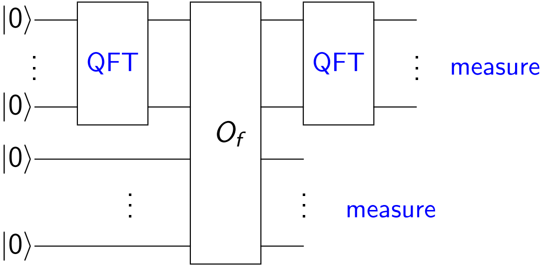
1. Shor's factoring algorithm
2. Grover's search algorithm
3. Ambainis's collision-finding algorithm
4. HHL algorithm for linear systems

Factoring

- Given $N = p \cdot q$, compute the prime factors p and q
- Fundamental mathematical problem since Antiquity
- Fundamental computational problem on $\log N$ bits
  $15 = 3 \times 5$
  $12140041 = 3413 \times 3557$
- Best known classical algorithms use time $2^{(\log N)^{\alpha}}$, where $\alpha = 1/2$ or $1/3$
- Its assumed computational hardness is basis of public-key cryptography (RSA)
- A quantum computer can break this,
  using Shor's efficient quantum factoring algorithm!

Overview of Shor's algorithm

- Classical reduction: choose random $x \in \{2, \ldots, N-1\}$.
  It suffices to find period r of $f(a) = x^a \bmod N$
- Shor uses the quantum Fourier transform for period-finding



- Overall complexity: roughly $(\log N)^2$ elementary gates

Reduction to period-finding

- Pick a random integer $x \in \{2, \ldots, N-1\}$, s.t. gcd(x, N)=1
- The sequence $x^0, x^1, x^2, x^3, \ldots \mod N$ cycles:

  has an unknown period r (min $r > 0$ s.t. $x^r \equiv 1 \mod N$)
- With probability $\geq 1/4$ (over the choice of x):
  r is even and $x^{r/2} \pm 1 \not\equiv 0 \mod N$
- Then:
  $$x^r = (x^{r/2})^2 \equiv 1 \mod N \iff$$
  $$(x^{r/2}+1)(x^{r/2}-1) \equiv 0 \mod N \iff$$
  $$(x^{r/2}+1)(x^{r/2}-1) = kN \text{ for some } k$$
- $x^{r/2}+1$ and $x^{r/2}-1$ each share a factor with N
- This factor of N can be extracted using gcd-algorithm

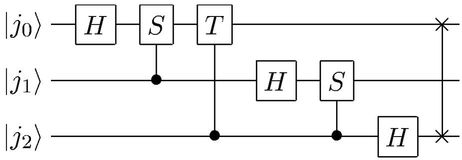
Quantum Fourier transform

- Fourier basis (dimension q): $|\chi_j\rangle = \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} e^{\frac{2\pi ijk}{q}} |k\rangle$

  Such a state is unentangled $|\chi_{j_0 j_1 j_2}\rangle =$

  $$\tfrac{1}{\sqrt{8}}(|0\rangle + e^{2\pi i 0.j_2}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.j_1 j_2}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.j_0 j_1 j_2}|1\rangle)$$

- Quantum Fourier Transform: $|j\rangle \mapsto |\chi_j\rangle$
- If $q = 2^\ell$, then can implement this with $O(\ell^2)$ gates.



- For Shor: choose $q = 2^\ell$ in $(N^2, 2N^2]$

Easy case for the analysis: $r|q$

1. Apply QFT to 1st register of $\underbrace{|0\ldots 0\rangle}_{\ell \text{ qubits}} \quad \underbrace{|0\ldots 0\rangle}_{\lceil \log N \text{ qubits} \rceil}$:

$$\frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle|0\rangle$$

2. Compute $f(a) = x^a \bmod N$ (by repeated squaring)

$$\frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle|x^a \bmod N\rangle$$

3. Observing 2nd register gives $|x^s \bmod N\rangle$ (random $s < r$)

   1st register collapses to superposition of

$$|s\rangle, |r+s\rangle, |2r+s\rangle, \ldots, |q-r+s\rangle$$

Easy case: $r|q$ (continued)

Recall: 1st register is in superposition $\sum_{j=0}^{q/r-1} |jr+s\rangle$

4. Apply QFT once more:

$$\sum_{j=0}^{q/r-1} \sum_{b=0}^{q-1} e^{2\pi i \frac{(jr+s)b}{q}} |b\rangle = \sum_{b=0}^{q-1} e^{2\pi i \frac{sb}{q}} \underbrace{\left(\sum_{j=0}^{q/r-1} \left(e^{2\pi i \frac{rb}{q}} \right)^j \right)}_{\text{geometric sum}} |b\rangle$$

Sum $\neq 0$ iff $e^{2\pi i \frac{rb}{q}} = 1$ iff $\frac{rb}{q}$ is an integer

Only the b that are multiples of $\frac{q}{r}$ have non-zero amplitude!

Easy case: $r|q$ (continued)

5. Observe 1st register: random multiple $b = c\frac{q}{r}$, $c \in [0, r)$:

$$\frac{b}{q} = \frac{c}{r}$$

- b and q are known; c and r are unknown
- c and r are coprime with probability $\geq 1/\log\log r$
- Then: we find r by writing $\frac{b}{q}$ in lowest terms
- Since we can find r, we can find prime factors of N !

Hard case ($r \not| q$) still works approximately: measurement gives b s.t. $\frac{b}{q} \approx \frac{c}{r}$; we can find r with some extra number theory

Summary for Shor's algorithm

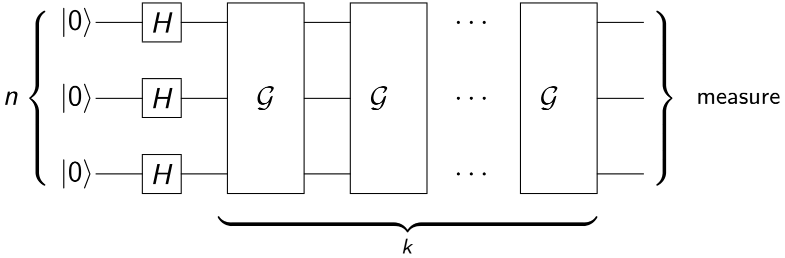
- Reduce factoring to finding the period r of modular exponentiation function $f(a) = x^a \bmod N$
- Use quantum Fourier transform to find a multiple of q/r, repeat a few times to find r
- Overall complexity:
  - QFT takes $O(\log q)^2 = O(\log N)^2$ elementary gates
  - Modular exponentiation: $\approx (\log N)^2 \log\log N$ gates; classical computation by repeated squaring (use Schönhage-Strassen algo for fast multiplication)
  - Everything repeated $O(\log\log N)$ times
  - Classical postprocessing takes $O(\log N)^2$ gates
- Roughly $(\log N)^2$ elementary gates in total

The search problem

- We want to search for some good item in an unordered N-element search space
- Model this as function $f : \{0,1\}^n \to \{0,1\}$ $(N = 2^n)$
  $f(x) = 1$ if x is a solution
- We can query f:
  $O_f : |x\rangle|0\rangle \mapsto |x\rangle|f(x)\rangle$
  or
  $O_f : |x\rangle \mapsto (-1)^{f(x)}|x\rangle$
- Goal: find a solution
- Classically this takes $O(N)$ steps (queries to f)
- Grover's algorithm does it in $O(\sqrt{N})$ steps

Grover's algorithm

- Apply Grover iteration $\mathcal{G}$ k times on uniform starting state



- Idea: each iteration moves amplitude towards solutions

The good state and the bad state

- Suppose there are t solutions
- Define "good" state and "bad" state:

$$|G\rangle = \frac{1}{\sqrt{t}} \sum_{x:f(x)=1} |x\rangle \qquad |B\rangle = \frac{1}{\sqrt{N-t}} \sum_{x:f(x)=0} |x\rangle$$

- Initial uniform state is $|U\rangle = \sin(\theta)|G\rangle + \cos(\theta)|B\rangle$ for $\theta = \arcsin(\sqrt{t/N})$
- All intermediate states will be in $\mathrm{span}\{|G\rangle, |B\rangle\}$
- Grover iteration is a rotation over angle 2θ

  so after k iterations the state is

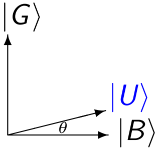
$$\sin((2k+1)\theta)|G\rangle + \cos((2k+1)\theta)|B\rangle$$

One Grover iteration: rotation by 2θ

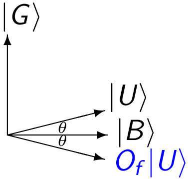
$\mathcal{G} = H^{\otimes n} R H^{\otimes n} \cdot O_f$, where R reflects through $|0^n\rangle$
This $\mathcal{G}$ is the product of two reflections:

1. O_f reflects through $|B\rangle$
2. $H^{\otimes n} R H^{\otimes n}$ reflects through $|U\rangle$

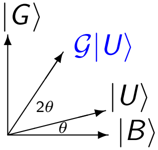
Starting state:



Reflect through $|B\rangle$:



Reflect through $|U\rangle$:



How many iterations do we need?

- Success probability after k iterations:

$$\sin^2((2k+1)\theta), \text{ with } \theta = \arcsin(\sqrt{t/N}) \approx \sqrt{t/N}$$

- If $k = \frac{\pi}{4\theta} - \frac{1}{2}$, then success probability is $\sin^2(\pi/2) = 1$
- Example: $t = N/4$ solutions $\Rightarrow k = 1$
- In general, round k to nearest integer (incurs small error)
- Query complexity is $k \approx \frac{\pi}{4}\sqrt{N/t}$

  This is optimal for a quantum algorithm!
- Gate complexity is $O(\sqrt{N/t} \log N)$

Summary for Grover's algorithm

- Quantum computers can search any N-element space with $t = \varepsilon N$ solutions, in $O(\sqrt{N/t}) = O(1/\sqrt{\varepsilon})$ iterations
  1. Set up uniform starting state $|U\rangle$
  2. Repeat the following $O(1/\sqrt{\varepsilon})$ times:
     2.1 Reflect through $|B\rangle$ (costs 1 query)
     2.2 Reflect through $|U\rangle$ (costs $O(\log N)$ gates)
  3. Measure final state to obtain an index i
- If we don't know $\varepsilon = t/N$, we can try different guesses, still find a solution with expected number of $O(1/\sqrt{\varepsilon})$ iterations
- The algorithm has a small error probability, but can be modified to error 0 *if* we know t exactly

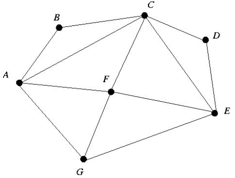
Application: Speed up NP problems

- Given a propositional formula $f(x_1, \dots, x_n)$
  Computable in time $\mathrm{poly}(n)$

  Question: is f satisfiable?

- This is a typical NP-complete problem
- Search space of $N = 2^n$ possibilities
- Classically: exhaustive search is the best we know.
  This takes about N steps
- Quantumly: Grover finds a satisfying assignment in
  $\sqrt{N} \cdot \mathrm{poly}(n)$ steps
- Because Grover is optimal, we believe that NP-hard problems cannot be efficiently computed by quantum algorithms

Classical random walks



- Explore a graph by moving to random neighbor in each step
- If G is d-regular and connected: normalized adjacency matrix has "spectral gap" $\delta \in (0,1)$. Starting from any vertex, $O(1/\delta)$ random walk steps produce uniform distribution
- Suppose an ε-fraction of the vertices are "marked" and we want to find such a marked vertex. Simple classical algorithm:
  1. Start at random vertex v (setup cost **S**)
  2. Do the following $O(1/\varepsilon)$ times:
     - 2.1 Check if v is marked (checking cost **C**)
     - 2.2 Rerandomize v by $O(1/\delta)$ RW steps (step cost **U**)

This finds marked item w.h.p. Cost is $\mathbf{S} + \frac{1}{\varepsilon}\left(\mathbf{C} + \frac{1}{\delta}\mathbf{U}\right)$

Quantum walks

- Quantum walk: walk in superposition over vertices (edges)
- Analogy with Grover's algorithm:
  $|G\rangle$ = uniform superposition over edges with marked endpoint
  $|B\rangle$ = uniform superposition over all other edges
  $|U\rangle = \sin(\theta)|G\rangle + \cos(\theta)|B\rangle$, $\theta = \arcsin(1/\sqrt{\varepsilon})$

1. Setup starting state $|U\rangle$ (setup cost **S**)
2. Repeat the following $O(1/\sqrt{\varepsilon})$ times:
   2.1 Reflect through $|B\rangle$ (checking cost **C**)
   2.2 Reflect through $|U\rangle$
   (can be implemented using $1/\sqrt{\delta}$ QW steps, each at cost **U**)
3. Measure and check that resulting vertex is marked.

Correctness analogous to Grover. Cost is $\mathbf{S} + \frac{1}{\sqrt{\varepsilon}}\left(\mathbf{C} + \frac{1}{\sqrt{\delta}}\mathbf{U}\right)$

Example: Ambainis's algorithm ('03)

Suppose we want to find a collision in $h : [n] \to \mathbb{N}$

- G =Johnson graph: the vertices are the sets $R \subseteq [n]$ of size r. Edge between sets R and R' if they differ in 1 element
- Fraction of vertices of G that contain collision: $\varepsilon \geq (r/n)^2$
- Known: spectral gap is $\delta \approx 1/r$
- With each vertex R, algorithm records $h(R)$;
  setup cost $\mathbf{S} = r$; checking cost $\mathbf{C} = 0$; update cost $\mathbf{U} = O(1)$
- Total cost: $\mathbf{S} + \frac{1}{\sqrt{\varepsilon}}\left(\mathbf{C} + \frac{1}{\sqrt{\delta}}\mathbf{U}\right) \overset{r=n^{2/3}}{=} O(n^{2/3})$
- Classically: $\Theta(n)$ f-evaluations needed

If h is 2-to-1: run on random set of $\sqrt{n}$ inputs (whp 1 collision) to get complexity $O(n^{1/3})$

Classically: $\Theta(\sqrt{n})$ f-evaluations, by birthday paradox

HHL algorithm for "solving" large linear systems

- Solving large linear systems $Ax = b$ is one of the most important problems in science and engineering.

  Goal: given matrix A and vector b, find vector x

- Harrow-Hassidim-Lloyd'09: "solves" this problem exponentially faster by preparing state $|x\rangle$ **IF** system is well-behaved:

  Assumptions

  (1) state $|b\rangle$ easy to prepare;

  (2) A is well-conditioned: $\lambda_{max}/\lambda_{min}$ not too big;

  (3) unitary operation e^{iA} is easy to apply (sparseness suffices)

How does the Harrow-Hassidim-Lloyd algorithm work?

- Input: Hermitian matrix $A \in \mathbb{R}^{N \times N}$ and vector $b \in \mathbb{R}^N$
  Goal: approximately prepare $|x\rangle$, where $Ax = b$
- Let $v_1, \ldots, v_N, \lambda_1, \ldots, \lambda_N$ be eigenvectors, eigenvalues of A
- HHL algorithm:
  1. Prepare quantum state $|b\rangle = \sum_{i=1}^{N} \beta_i |v_i\rangle$
     NB: applying A^{-1} corresponds to multiplying with λ_i^{-1}
  2. Use eigenvalue estimation: $\sum_{i=1}^{N} \beta_i |v_i\rangle |\lambda_i\rangle$
  3. Make new qubit $\sum_{i=1}^{N} \beta_i |v_i\rangle |\lambda_i\rangle \left(\lambda_i^{-1}|0\rangle + \sqrt{1 - \lambda_i^{-2}}|1\rangle \right)$
  4. Uncompute $|\lambda_i\rangle$ by inverting eigenvalue estimation
  5. Amplify the $|0\rangle$-part to end with $\sum_{i=1}^{N} \beta_i \lambda_i^{-1} |v_i\rangle = |x\rangle$

What else can a quantum computer do?

- Similar to Shor: discrete logarithm, solve Pell's equation, compute properties of number fields, ...
- Similar to Grover: maximum-finding, approximate counting, shortest paths in graphs, minimum spanning trees, ...
- Similar to quantum walks: finding small subgraphs, matrix-product verification, junta-testing, backtracking, ...
- Similar to HHL: quantum machine learning, principal component analysis, recommendation systems, ...
- Efficiently simulating quantum-mechanical systems.

  Could be very important for drug design, material sciences...

What quantum algorithms *cannot* do

- You can simulate every quantum algorithm with an exponentially slower classical computer

  This implies that the set of *computable* problems doesn't change: Church-Turing thesis remains intact

- For many problems we can show that quantum computers give no significant speed-up

  or at most a quadratic speed-up (e.g., Grover is optimal)

- NP-complete problems form a famous and important class of hard computational problems: satisfiability, Traveling Salesman Problem, protein folding,...

  Conjectured: quantum computers can't efficiently solve them

Conclusion

- Quantum mechanics is the best physical theory we have
- Fundamentally different from classical physics:
  superposition, interference, entanglement
- Quantum algorithms use these non-classical effects to solve some problems much faster
- We saw 4 important examples:
  1. Shor's factoring algorithm
  2. Grover's search algorithm
  3. Ambainis's collision-finding algorithm
  4. HHL algorithm for linear systems

Much more left to be discovered...

Phase estimation

- Suppose we can apply U and are given one of its eigenvectors $|v\rangle$ as a quantum state. Goal: learn eigenvalue $e^{2\pi i\theta}$
  Suppose phase $\theta = 0.\theta_1 \ldots \theta_\ell$ has ℓ bits of precision
- Remember QFT: $|j\rangle \mapsto |\chi_j\rangle = \frac{1}{\sqrt{2^\ell}} \sum_{k=0}^{2^\ell-1} e^{\frac{2\pi ijk}{2^\ell}} |k\rangle$
- Phase estimation algorithm:
  1. Start with $|0^\ell\rangle|v\rangle$
  2. Apply $H^{\otimes\ell}$: $\frac{1}{\sqrt{2^\ell}} \sum_{k\in\{0,1\}^\ell} |k\rangle|v\rangle$
  3. Conditioned on 1st register, apply U^k to 2nd register:
  $$\frac{1}{\sqrt{2^\ell}} \sum_{k\in\{0,1\}^\ell} |k\rangle e^{2\pi i\theta k}|v\rangle = \frac{1}{\sqrt{2^\ell}} \sum_{k\in\{0,1\}^\ell} e^{2\pi i\theta k}|k\rangle|v\rangle$$
  4. Inverse QFT on first register gives $j = \theta 2^\ell = \theta_1 \ldots \theta_\ell$
- With $O(1/\varepsilon)$ applications of U: ε-error approximation of θ