KU LEUVEN

MQ Signatures for PKI

June 2017

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Ward Beullens, Bart Preneel

New Hope Key Exchange

- Post-Quantum KX based on RLWE
- USENIX 2016

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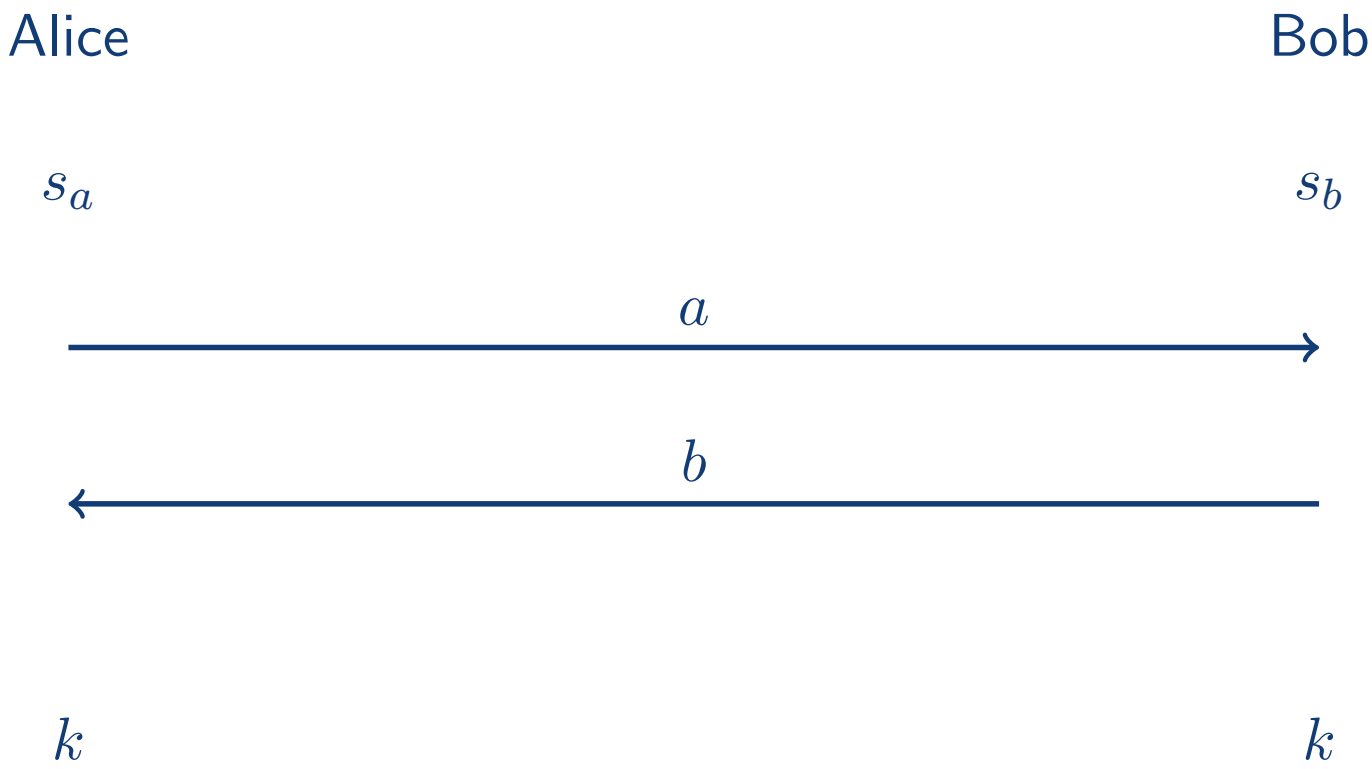
Post-Quantum Key Exchange



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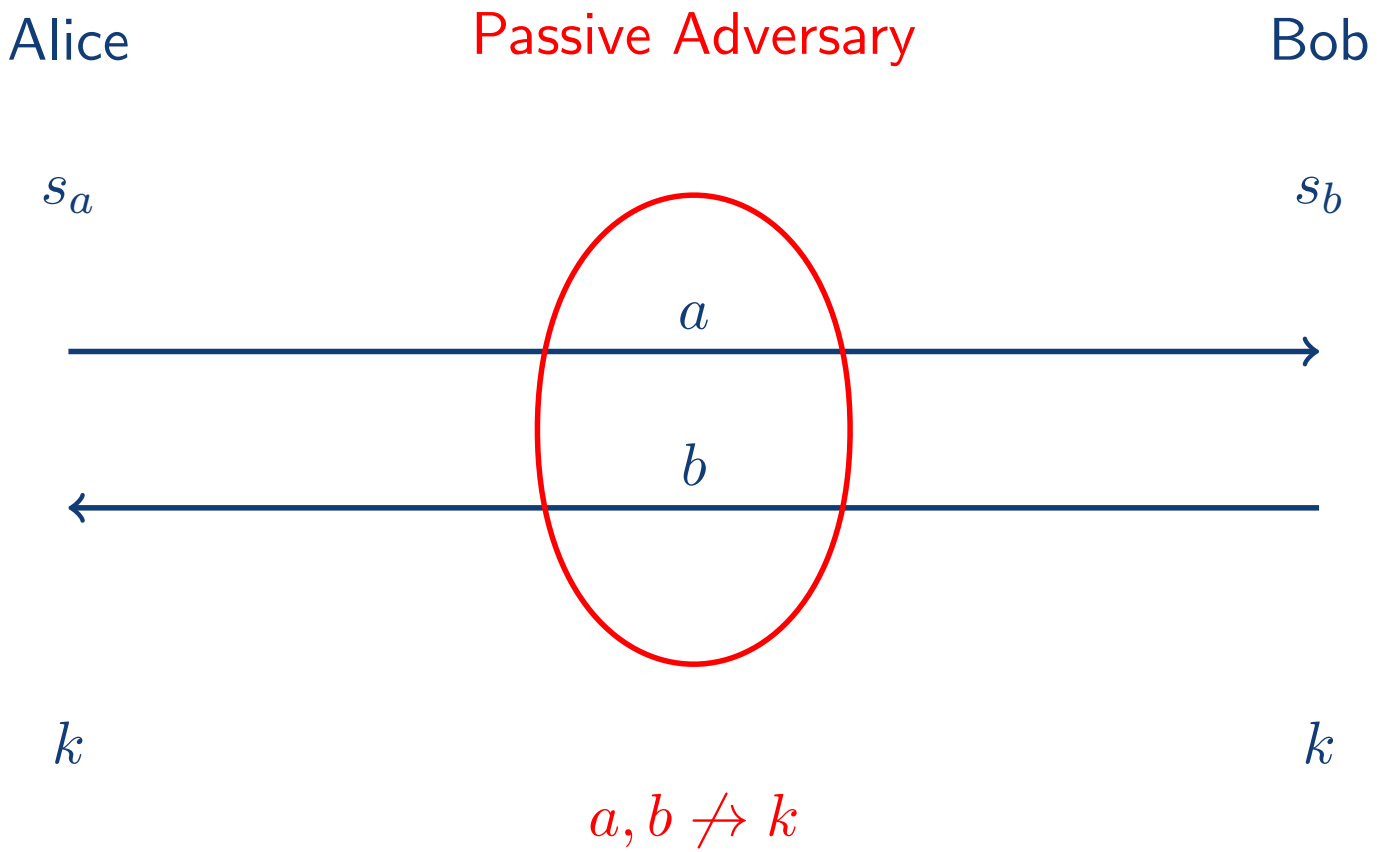
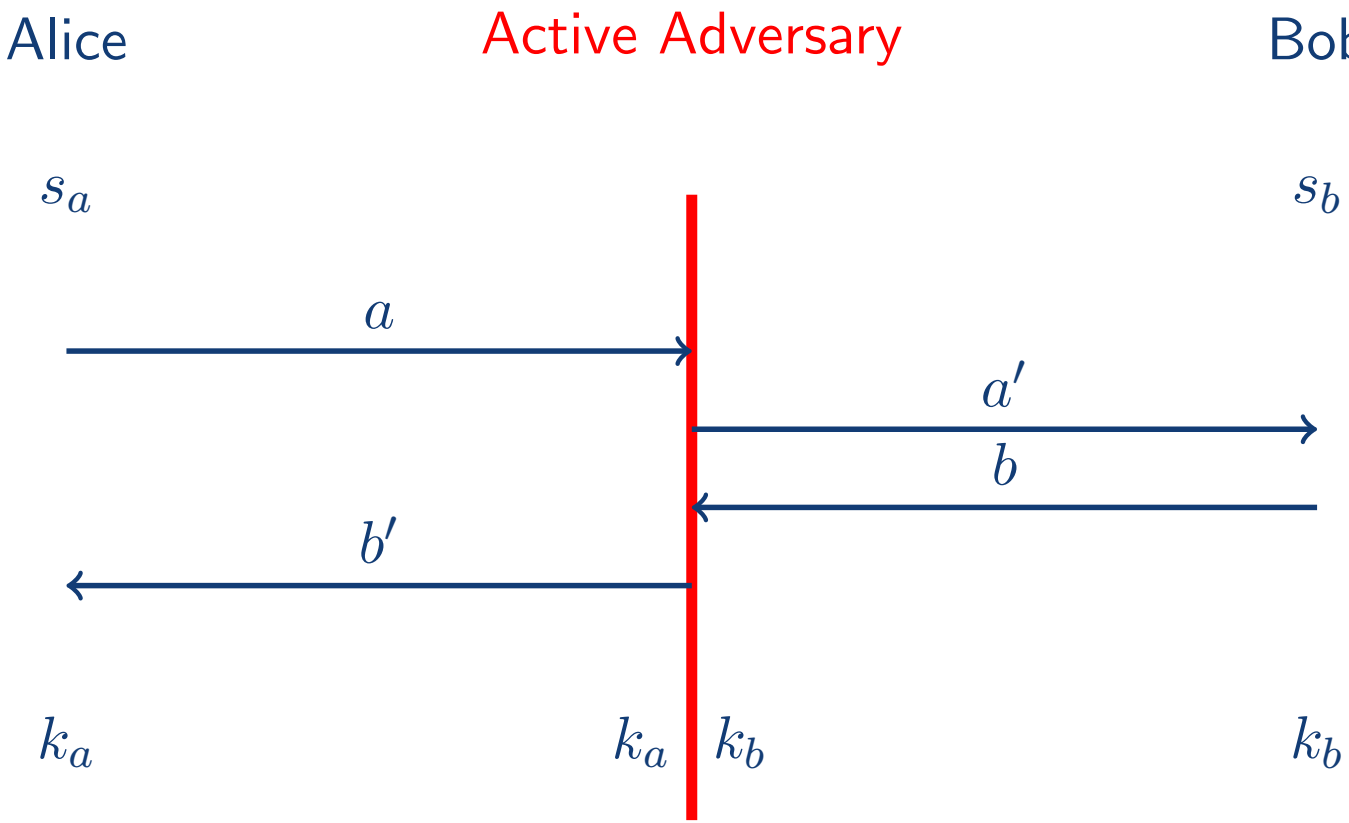


Post-Quantum Key Exchange

Alice | Active Adversary | Bob

s_a s_b

Post-Quantum Key Exchange



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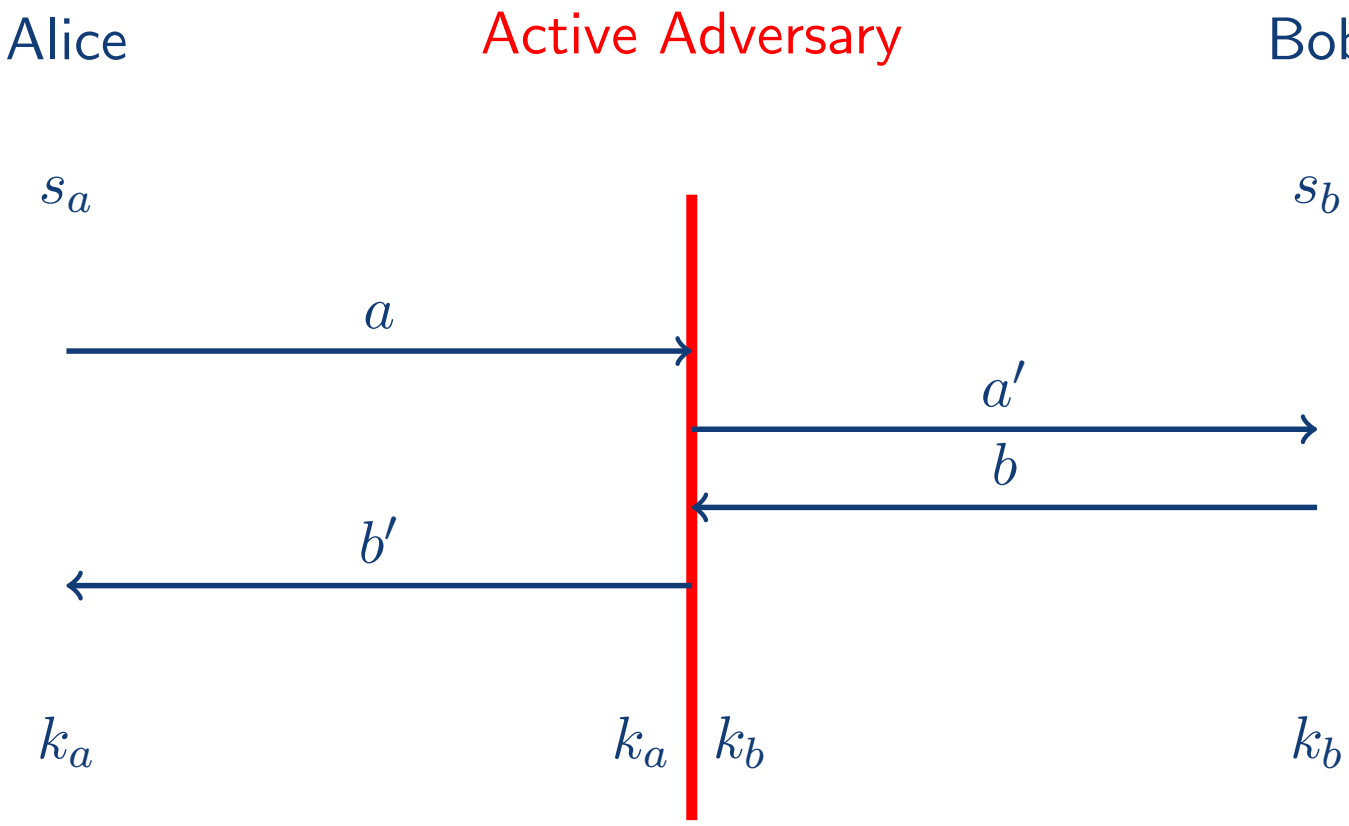
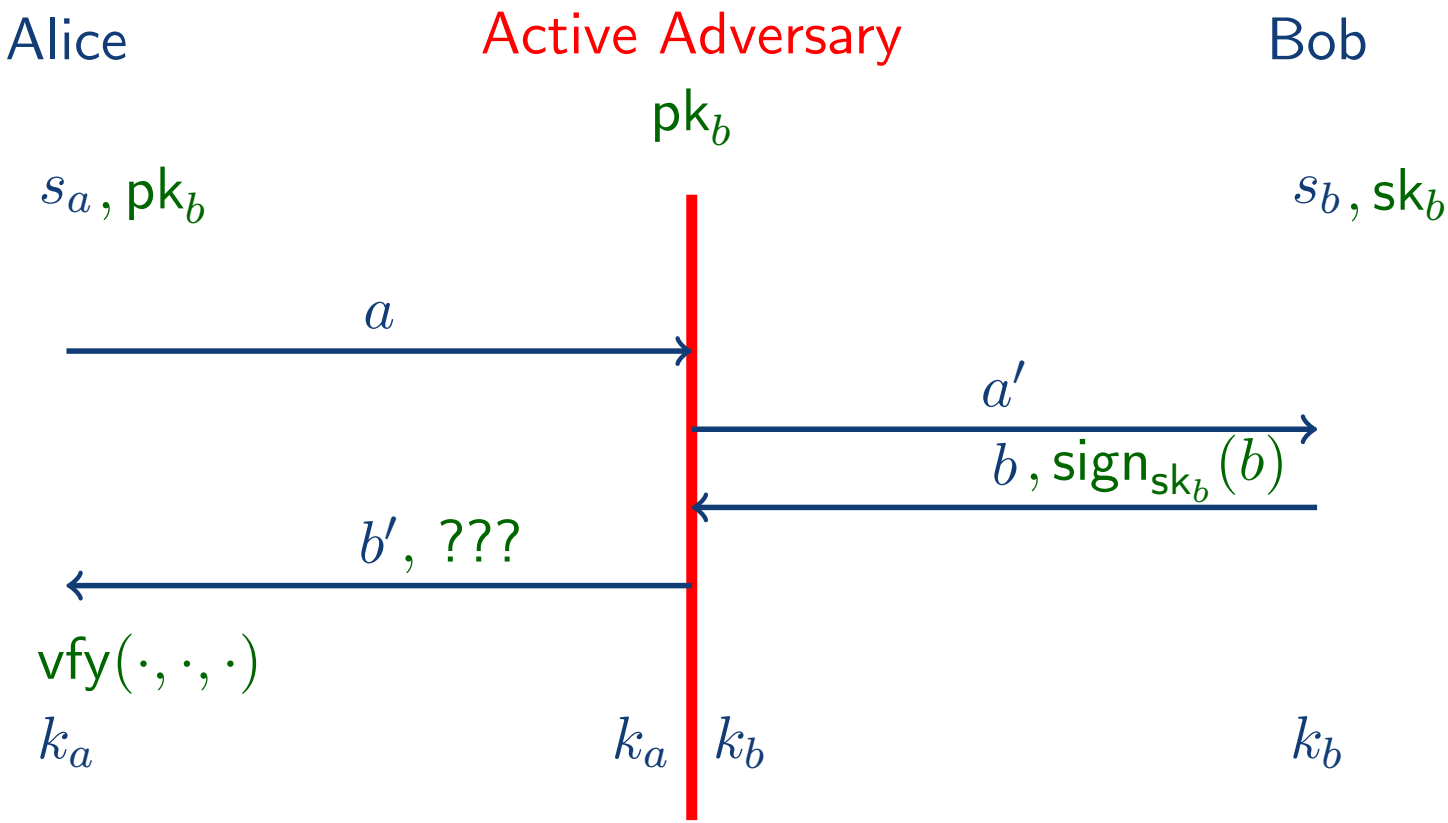


How to kill MitM?

Post-Quantum Key Exchange



How to kill MitM? Signatures, of course!

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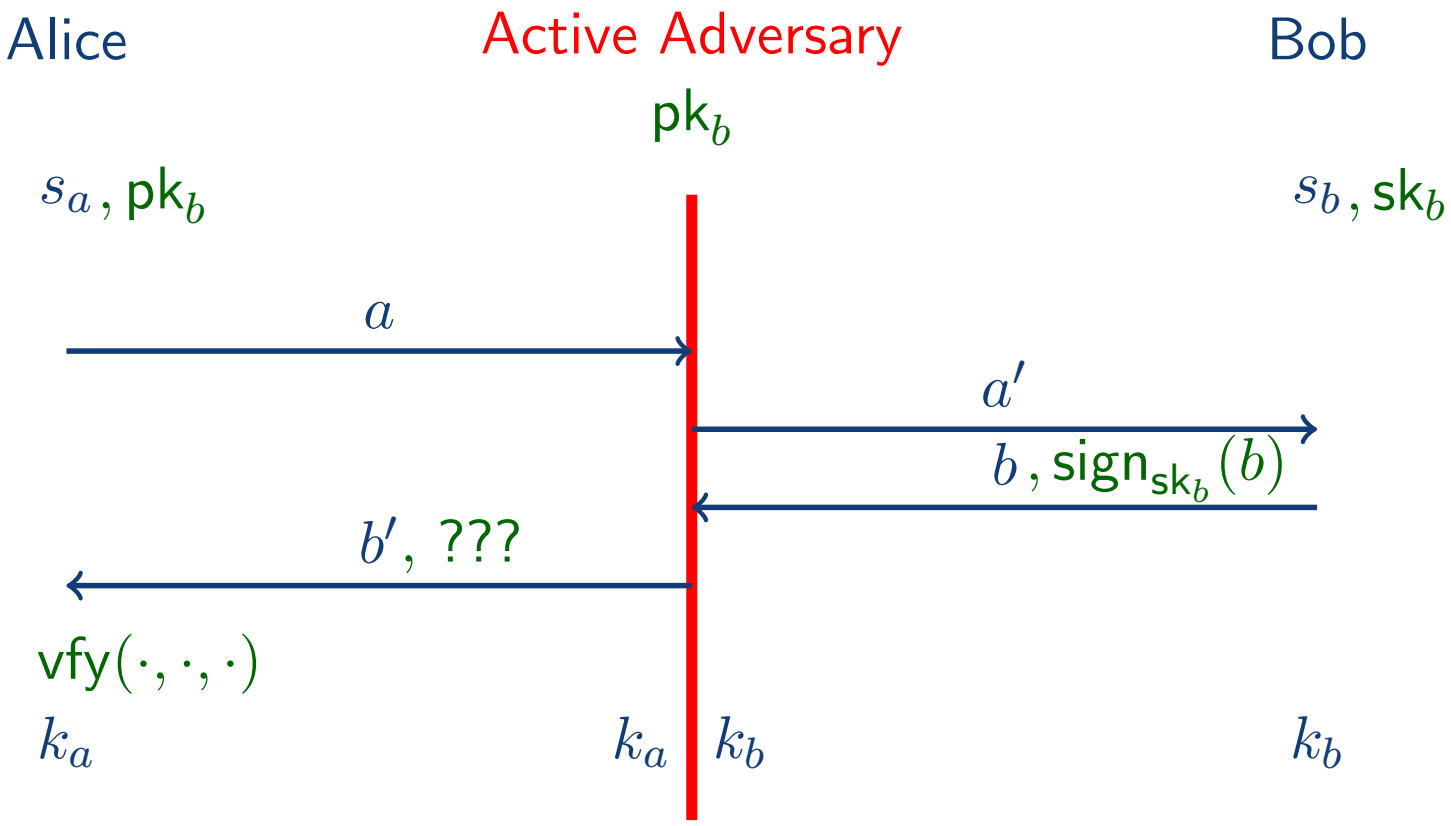

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Post-Quantum

Public Key Infrastructure (PKI)

Alice

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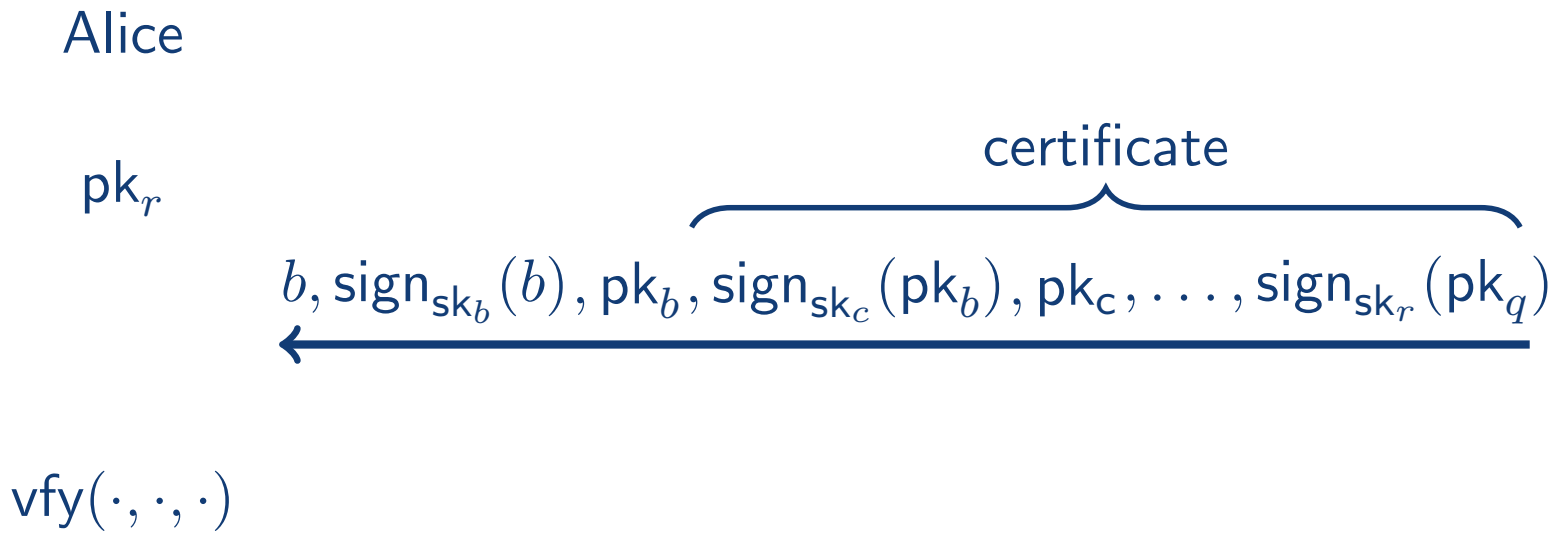

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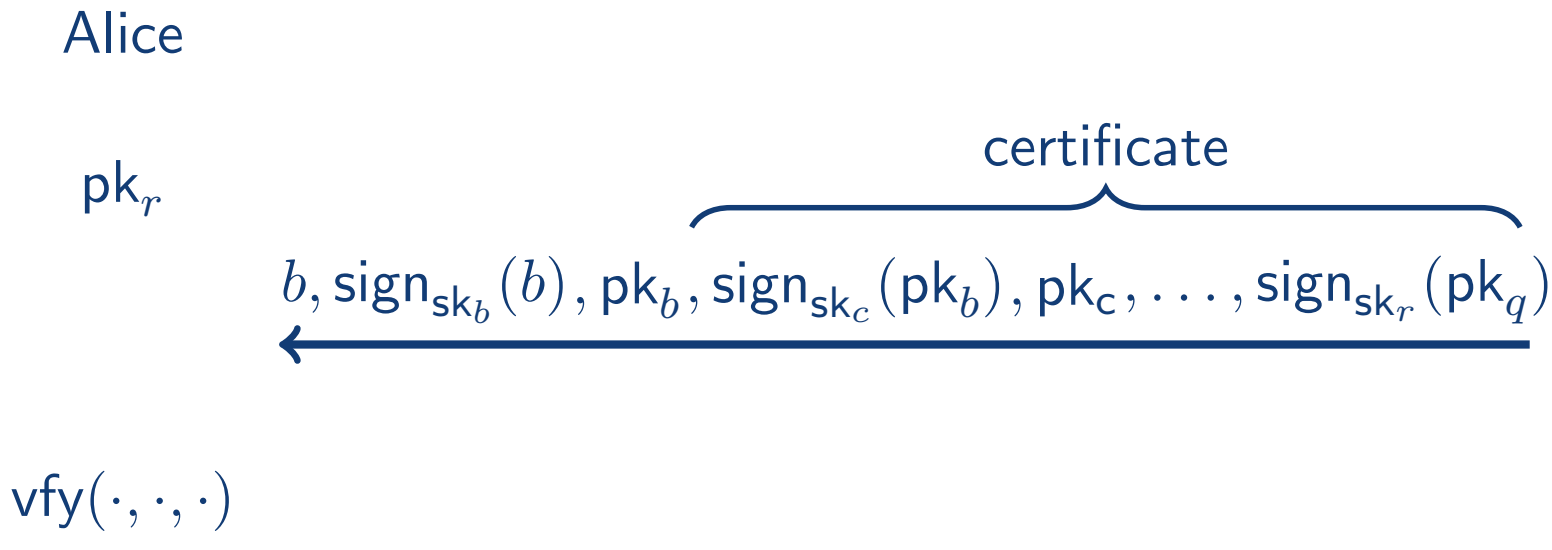

desirable properties

acceptable drawbacks

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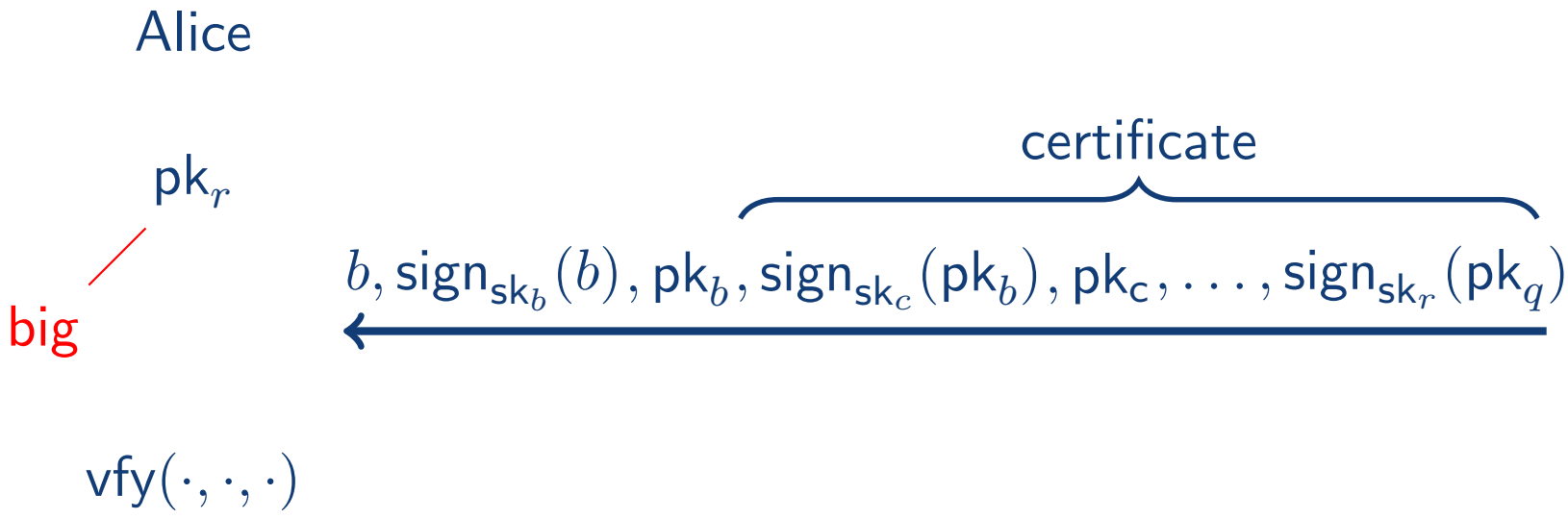

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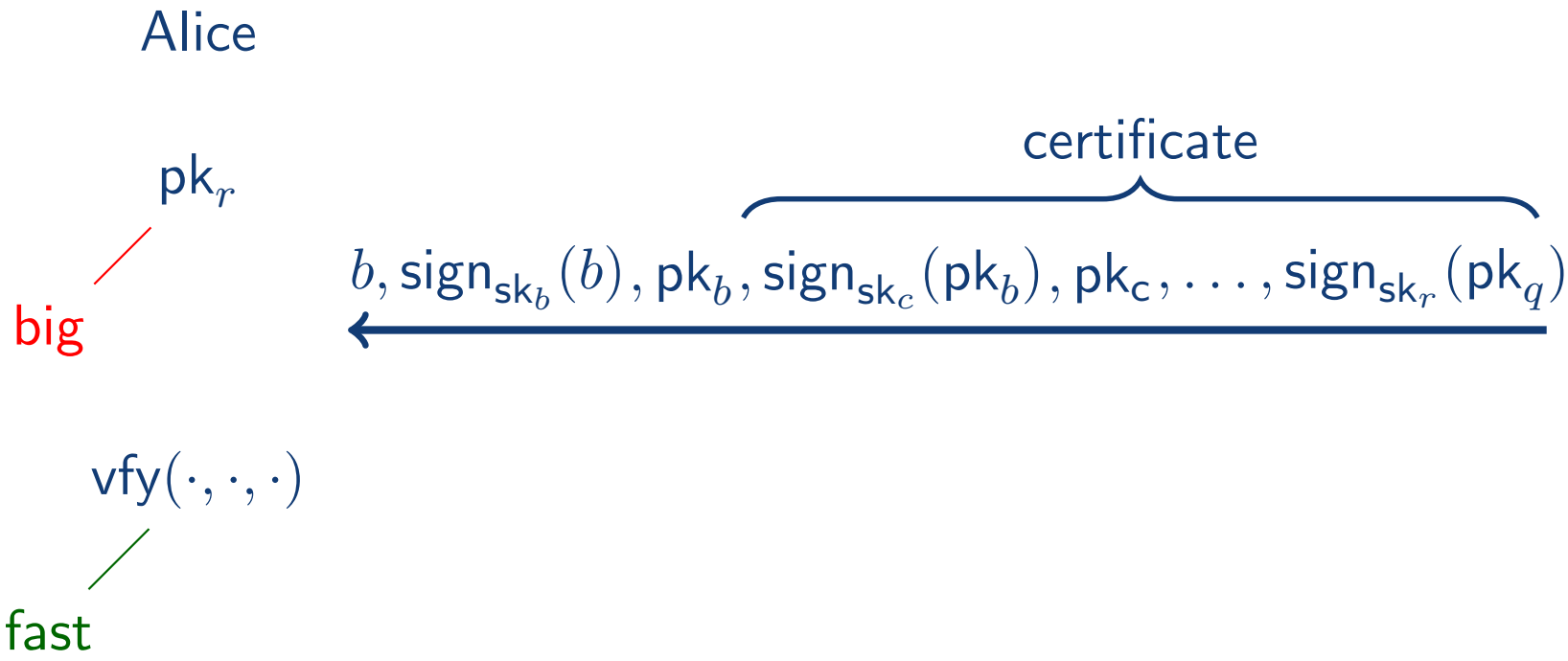

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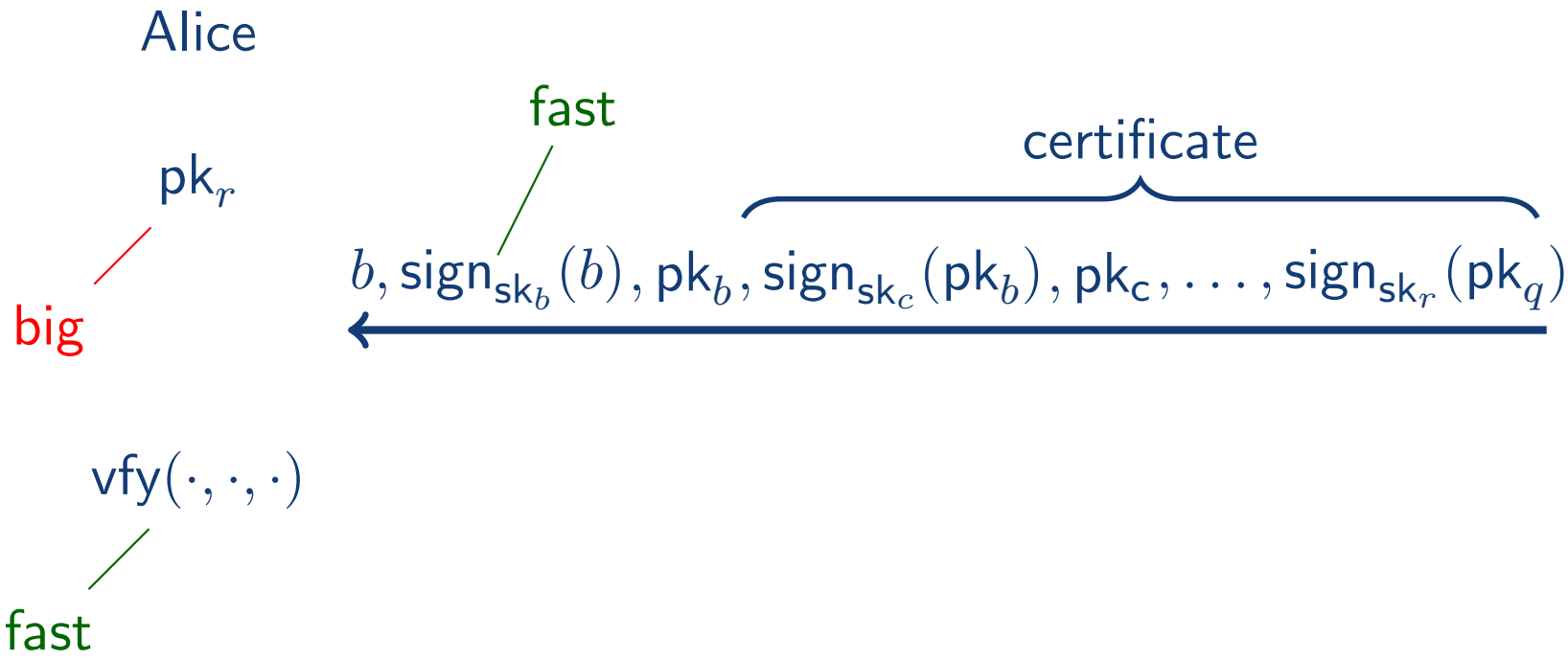

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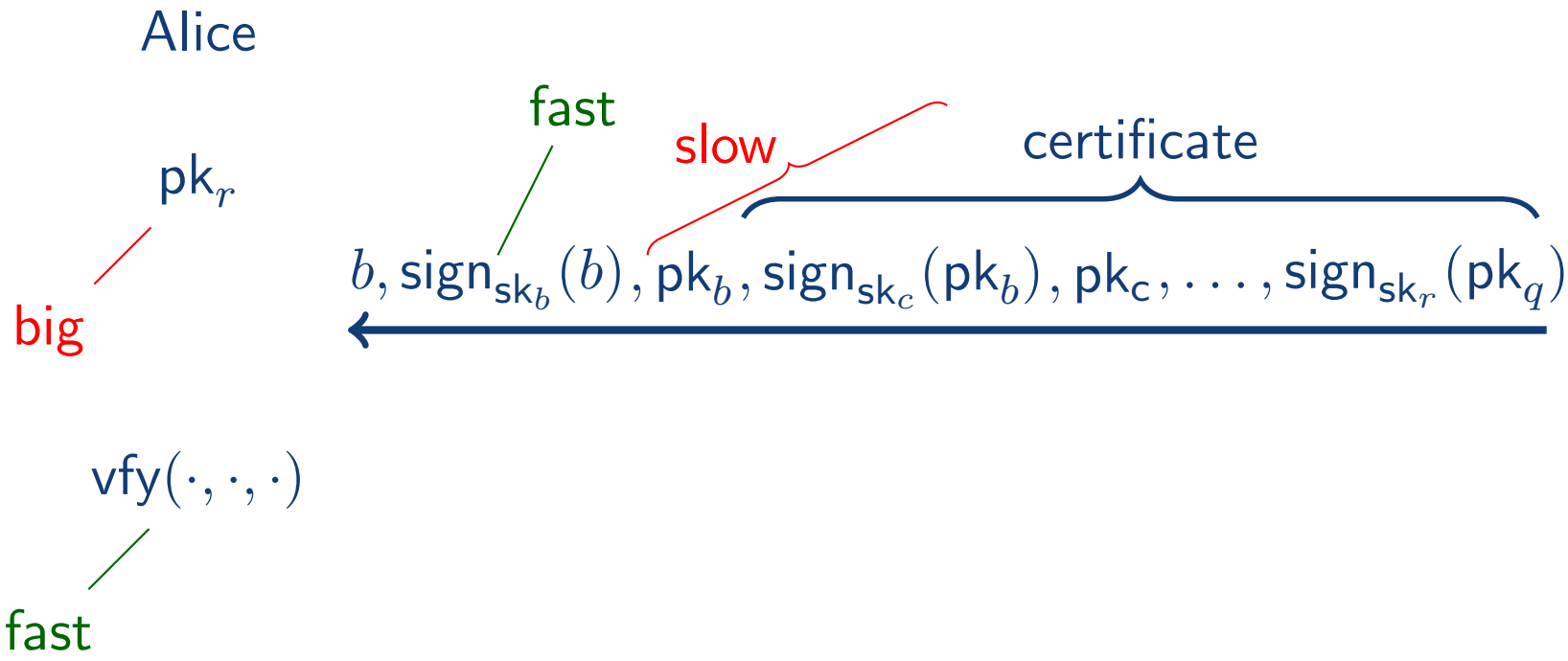

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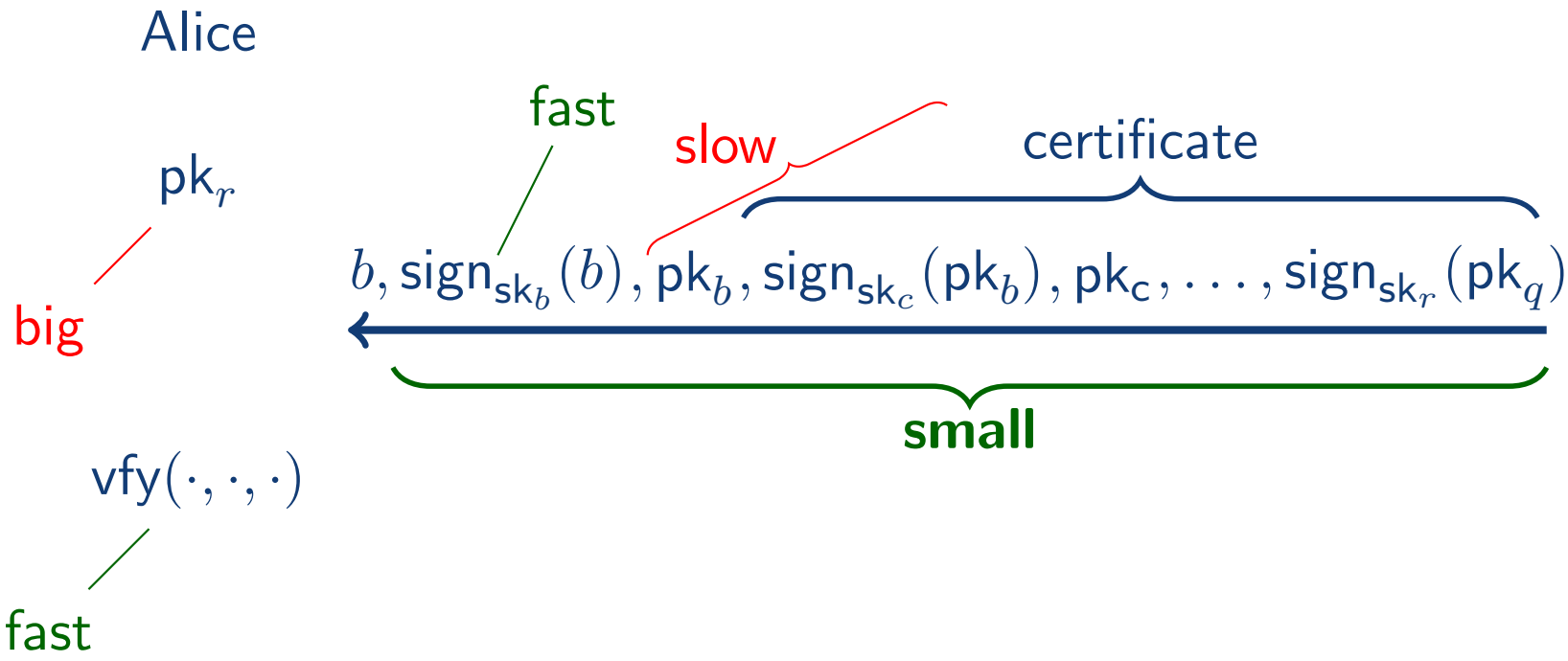

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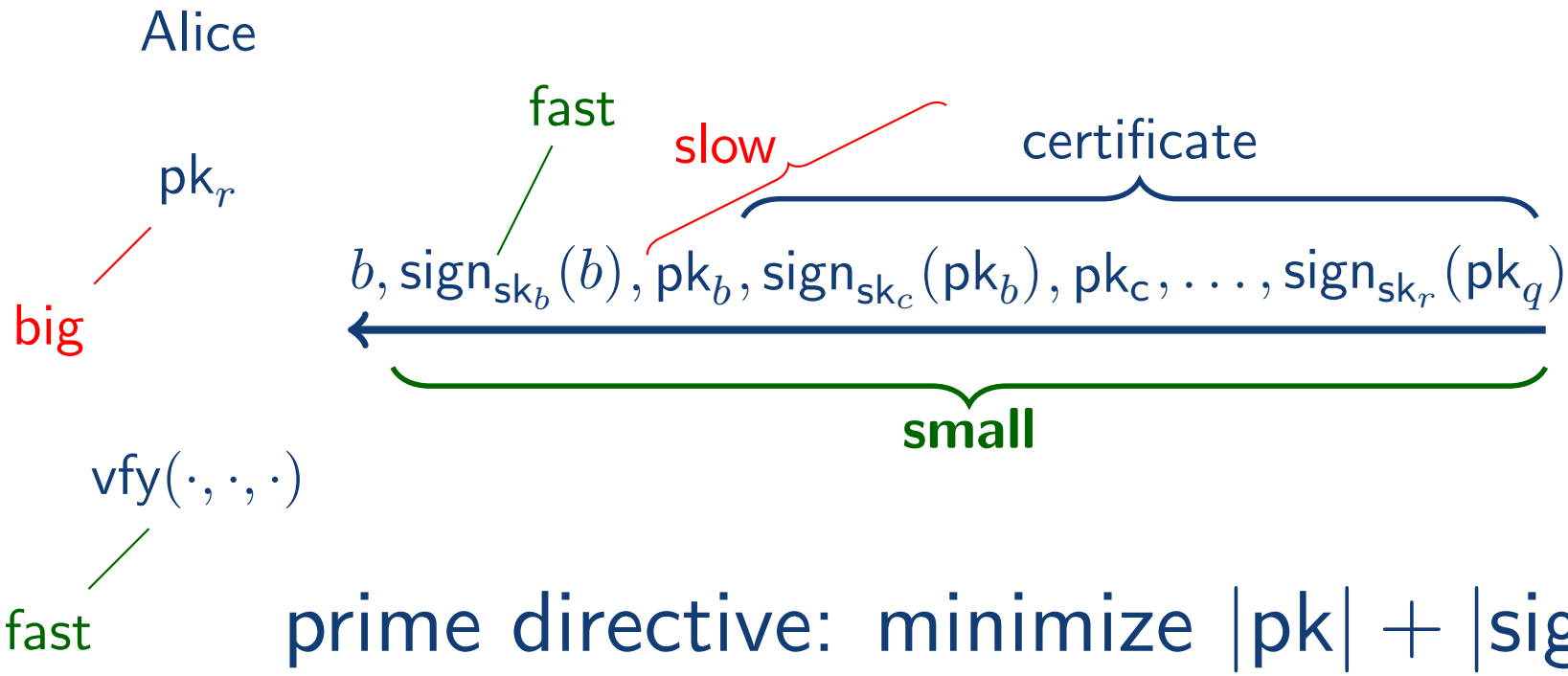
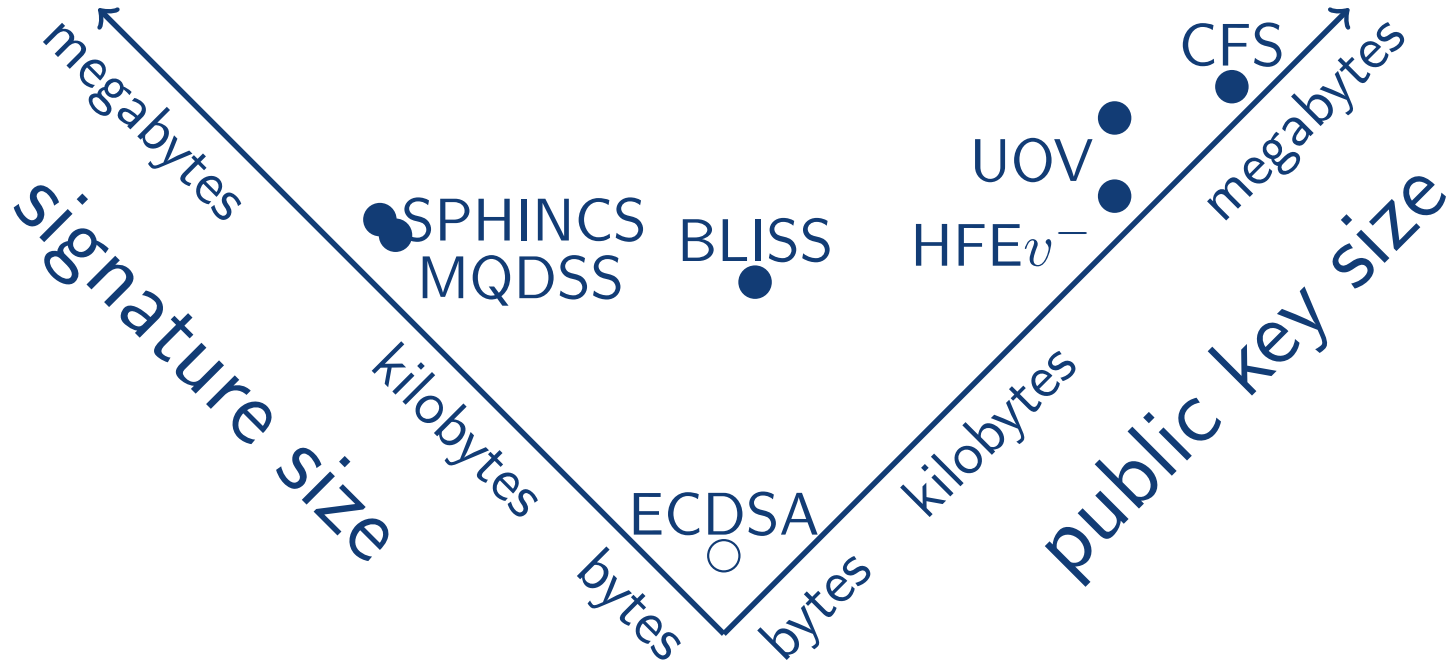


prime directive: minimize $|\mathsf{pk}| + |\mathsf{sig}|$

desirable properties

acceptable drawbacks

Post-Quantum Signature Schemes



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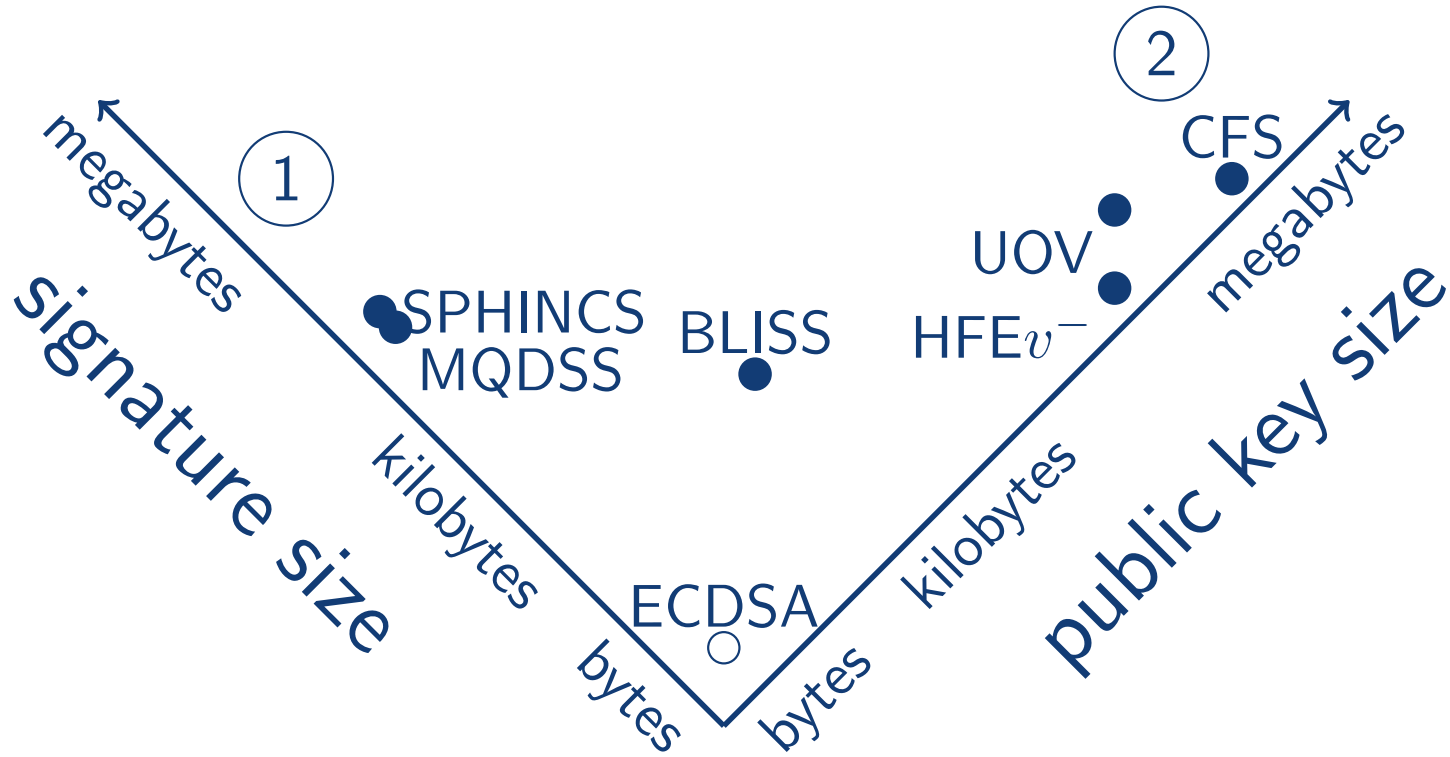

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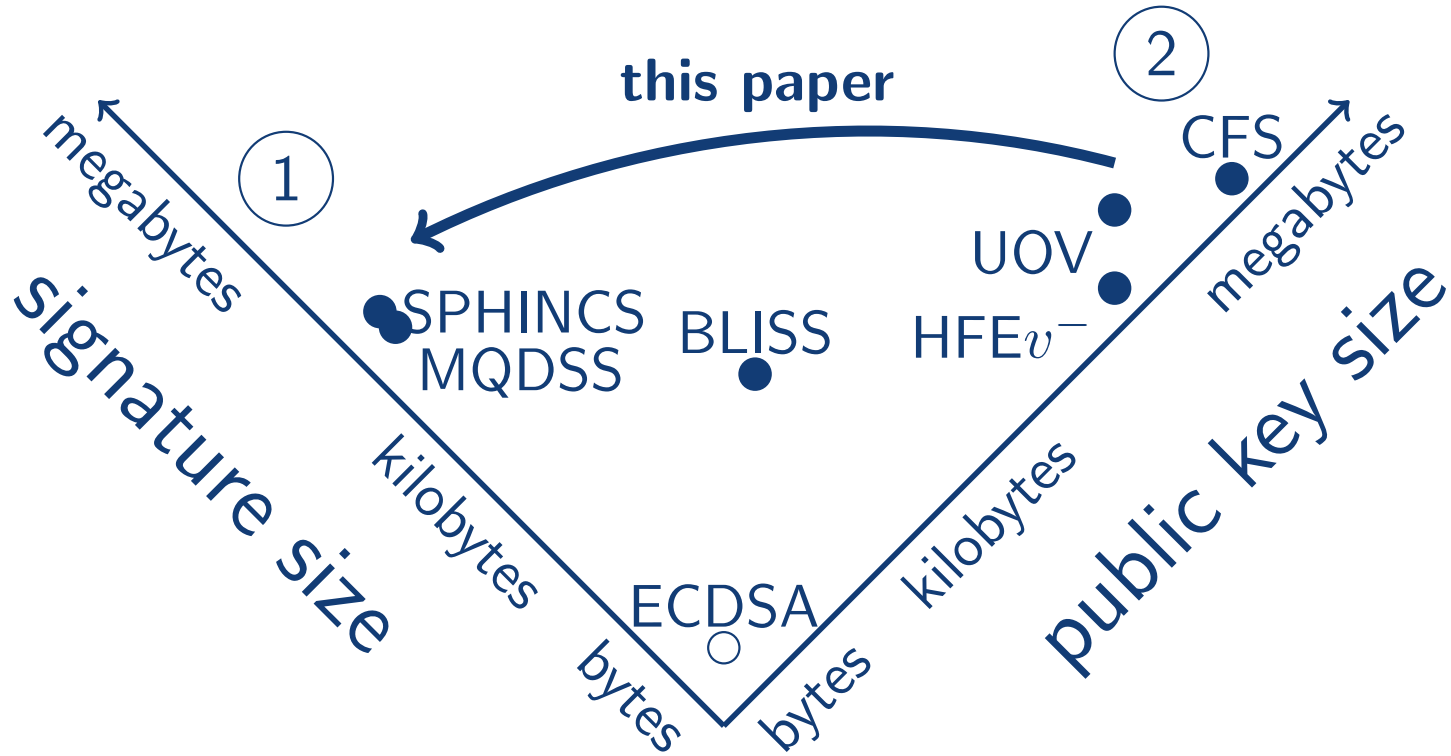

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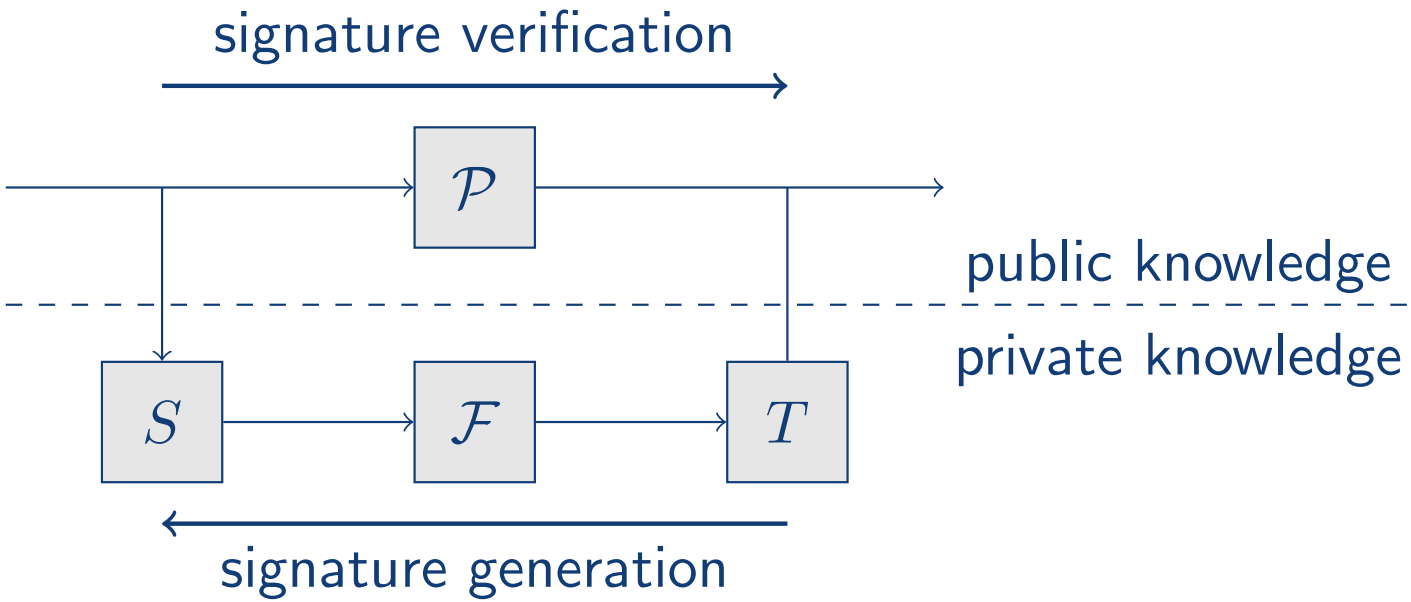
MQ Signature Schemes



$\mathcal{P}, \mathcal{F} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ $\qquad$ $\mathbf{s} \in \mathbb{F}_q^n$

$T, S \in \mathsf{GL}(\mathbb{F}_q)$ $\qquad$ $\mathsf{vfy} : \mathcal{P}(\mathbf{s}) \stackrel{?}{=} \mathsf{H}(d)$

Transformation

Step 1: replace $\mathcal{P}(\mathbf{s}) \stackrel{?}{=} \mathsf{H}(d)$ with $\mathbf{t}\mathcal{P}(\mathbf{s}) \stackrel{?}{=} \mathbf{t}\mathsf{H}(d)$ for $\mathbf{t} \stackrel{\$}{\leftarrow} \mathbb{F}_q^{\alpha \times m}$

determine $\mathbf{t} = \mathsf{H}(\mathbf{s})$

transmit $\mathcal{R}(\mathbf{x}) = \mathbf{t}\mathcal{P}(\mathbf{x})$ with $\mathbf{s}$ as part of signature

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2b. verify that $\mathbf{t}\hat{\mathcal{P}}(z) = \hat{\mathcal{R}}(z)$ instead of $\mathbf{t}\mathcal{P}(\mathbf{x}) = \mathcal{R}(\mathbf{x})$

$$\begin{array}{ccc} \mathcal{P}(\mathbf{x}) & \xrightarrow{\text{MAC}} & \hat{\mathcal{P}}(z) \\ \downarrow \mathbf{t} & & \downarrow \mathbf{t} \\ \mathbf{t}\mathcal{P}(\mathbf{x}) & \xrightarrow{\text{MAC}} & \mathbf{t}\hat{\mathcal{P}}(z) \end{array}$$

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2c. verify $\mathbf{t}\hat{\mathcal{P}}(z_i) = \hat{\mathcal{R}}(z_i)$ in only ϑ randomly chosen points

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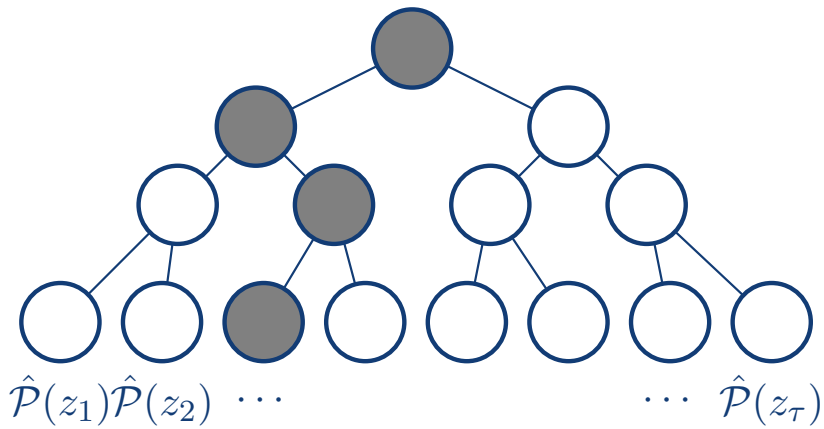
2d. Merkleize all τ evaluations $\hat{\mathcal{P}}(z_i)$

new signature: $(\mathbf{s}, \mathcal{R}(\mathbf{x}), \textit{Merkle paths})$ new public key: $\textit{Merkle root}$

Merkle Tree


$\hat{\mathcal{P}}(z_1)\hat{\mathcal{P}}(z_2)$ $\cdots$
$\cdots$ $\hat{\mathcal{P}}(z_\tau)$

Provable Security

$$
\begin{aligned}
\mathsf{InSec}^{\mathsf{EUF-CMA}}_{\mathsf{NEW}}(t, Q) \leq \; & \mathsf{InSec}^{\mathsf{EUF-CMA}}_{\mathsf{ORIGINAL}}(t + O(Q), Q) \\
+ \quad & (2\tau - 1)\frac{Q+1}{2^{\kappa}} \\
+ \quad & \left(\frac{n(n+1)}{2\tau}\right)^{\vartheta}(Q+1) \\
+ \quad & q^{-\alpha}(Q+1)
\end{aligned}
$$

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+\quad & \left(\tfrac{n(n+1)}{2\tau}\right)^{\vartheta}(Q+1) && \Big\} \text{ MAC polynomials} \\
+\quad & q^{-\alpha}(Q+1) && \Big\} \text{ lucky s}
\end{aligned}
$$

Provable Security

... in the QROM

$$\mathsf{InSec}^{\mathsf{EUF-CMA}}_{\mathsf{NEW}}(t,\hat{Q}) \leq \mathsf{InSec}^{\mathsf{EUF-CMA}}_{\mathsf{ORIGINAL}}(t+O(\hat{Q}),\hat{Q}) \quad \} \text{ original scheme}$$

$$+\Theta\left((2\tau-1)\frac{(\hat{Q}+1)^2}{2^\kappa}\right) \quad \} \text{ Merkle tree}$$

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Example Parameters

scheme	parameters	sec. lvl.	$\lvert\mathsf{pk}\rvert$	$\lvert s\rvert$
UOVrand	$q = 256, n = 135, m = 45$	128	45.5 kB	1080
transformed	$\alpha = 16, \vartheta = 12, \tau = 2^{20}$	128	256 bits	21.3 kB
UOVrand	$q = 256, n = 210, m = 70$	192	169.9 kB	1 680 bits
transformed	$\alpha = 24, \vartheta = 19, \tau = 2^{20}$	192	384 bits	70.4 kB
UOVrand	$q = 256, n = 285, m = 95$	256	423.0 kB	2 280 bits
transformed	$\alpha = 32, \vartheta = 28, \tau = 2^{20}$	256	512 bits	166.3 kB

Improvement

- idea: use *multiple* signatures
- $\mathbf{s}_1, \ldots, \mathbf{s}_\sigma$ such that $\mathcal{P}(\mathbf{s}_i) = \mathsf{H}(d\|i)$

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- $\mathbf{s}_1, \ldots, \mathbf{s}_\sigma$ such that $\mathcal{P}(\mathbf{s}_i) = \mathsf{H}(d \| i)$
- ... reduce $\alpha \quad \longrightarrow$ fewer polynomials in $\mathcal{R}$
- $|\mathbf{s}_i| = n \log_2 q$ whereas $|\mathcal{R}_i(\mathbf{x})| = \frac{n(n+1)}{2} \log_2 q$
- $q^\alpha > 2^\kappa$ becomes $q^{\alpha\sigma} > 2^\kappa$

Security Proof Fails

$$
\begin{aligned}
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& + \Theta\left((2\tau-1)\tfrac{(\hat{Q}+1)^2}{2^\kappa}\right) && \text{Merkle tree}\\
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& + \Theta\left(q^{-\sigma\alpha}(\hat{Q}+1)^2\right) && \text{lucky s}
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$$

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Low-Dim Errors

- find $\mathbf{s}_1, \ldots, \mathbf{s}_\sigma$ such that $\forall i \,.\, \mathbf{t}\mathcal{P}(\mathbf{s}_i) = \mathbf{t}\mathsf{H}(d\|i)$
- model $\mathbf{t} \leftarrow \mathsf{H}(d\|\mathbf{s}_1\| \cdots \|\mathbf{s}_\sigma)$ as $\mathbf{t} \xleftarrow{\$} \mathbb{F}_q^{\alpha \times m}$

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- works if $\mathbf{0} \neq \mathcal{P}(\mathbf{s}_i) - \mathsf{H}(d\|i) \in \ker \mathbf{t}$
- *error space is low-dim* $\implies$ *high success probability*

AMQ Problem

Definition

AMQ Problem. (Approximate Multivariate Quadratic)
Given: $\mathcal{P} : \mathbb{F}_q^n \to \mathbb{F}_q^m; \mathbf{y}_1, \ldots, \mathbf{y}_\sigma \in \mathbb{F}_q^m$
Find: $\mathbf{x}_1, \ldots, \mathbf{x}_\sigma \in \mathbb{F}_q^n$
Such that: $\dim \langle \{\mathcal{P}(\mathbf{x}_i) - \mathbf{y}_i\}_i \rangle \leq r$

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Definition

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- exhaustive search: $O(q^{m-r})$; Grover: $O(q^{(m-r)/2})$
- $\mathsf{AMQ}[m, n, \sigma, r] \leq \sigma \cdot \mathsf{MQ}[m - r, n]$
- $\mathsf{AMQ}[m, n, \sigma, r] \leq \mathsf{AMQ}[m, n, \sigma + 1, r]$ (gets harder with σ)
- $\mathsf{AMQ}[m, n, \sigma, r + 1] \leq \mathsf{AMQ}[m, n, \sigma, r]$ (gets easier with r)
- $\mathsf{AMQ}[m, n, \sigma = 1, r = 0] = \mathsf{MQ}[m, n]$

Example Parameters

scheme	parameters	sec. lvl.	|pk|	$\|s\|$
original HFEv^-	$q = 2, n = 98, m = 90$	80	56.8 kB	98 bits
transformed	$\alpha = 1, \sigma = 80, \vartheta = 7, \tau = 2^{20}$	80 ?	80 bits	4.4 kB
original HFEv^-	$q = 2, n = 133, m = 123$	120	139.2 kB	123 bits
transformed	$\alpha = 1, \sigma = 120, \vartheta = 11, \tau = 2^{20}$	120 ?	120 bits	9.4 kB
original HFEv^-	$q = 4, n = 141, m = 129$	128 (PQ)	288.4 kB	258 bits
transformed	$\alpha = 1, \sigma = 64, \vartheta = 13, \tau = 2^{20}$	128 ? (PQ)	256 bits	16.5 kB