An Updated Security Analysis of PFLASH

Ryann Cartor[1] & **Daniel Smith-Tone**[1,2]

[1]University of Louisville
[2]National Institute of Standards and Technology

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A system of (generally structured) nonlinear equations. Typically a quadratic trapdoor function.

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Verification

One verifies by evaluating the public key at the signature value.

Hiding Structure

Polynomial Morphisms

Let $f, g : \mathbb{F}^n \to \mathbb{F}^m$ be polynomial functions. A *polynomial morphism* between f and g is a pair of affine transformations $T \in M_m(\mathbb{F})$ and $U \in M_n(\mathbb{F})$ such that

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Isomorphism of Polynomials

If T and U are nonsingular, then the pair (T, U) is an *isomorphism of polynomials*. Further, if T is the identity, then U is called a one-sided isomorphism between f and g.

Big Field Schemes

Construct an extension field $\mathbb{K}$ of $\mathbb{F}$. One may think of the extension as a commutative $\mathbb{F}$-algebra that happens to be a field. One utilizes the multiplication in $\mathbb{K}$ to construct an invertible map.

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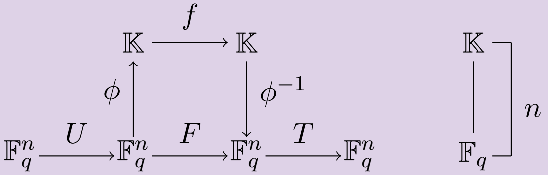
Butterfly Construction



C^* Scheme

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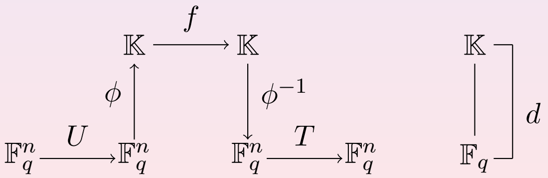
- Thus the public key is given by $P = T \circ \phi^{-1} \circ f \circ \phi \circ U$.



Minus and projection modifiers

Minus modifier: delete r of the n public key equations.

Projection modifier: fix the value of d input variables.

PFLASH

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The public key is given by:

$$P = \pi_r \circ T \circ \phi^{-1} \circ f \circ \phi \circ U \circ \pi_d,$$

where π_r and π_d are codimension r and d affine projections, respectively.

Discrete Differential

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$Df(a,x) = f(a+x) - f(a) - f(x) + f(0)$.

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For any quadratic function f, Df is bilinear.

Differential Symmetry

A function $f : \mathbb{K} \to \mathbb{K}$ has a **differential symmetry** if there exists a pair of $\mathbb{F}_q$-linear maps $M, \Lambda_M : \mathbb{K} \to \mathbb{K}$ such that

$$Df(Ma, x) + Df(a, Mx) = \Lambda_M Df(a, x)$$

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Thus, differential symmetry can be exploited to remove the minus modifier when there is a space of nontrivial (i.e. non-scalar) linear maps inducing a differential symmetry on the central map.

Differential analysis of pC^*

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- As shown in [Chen et al.,2015], we may assume the projection mapping is tied to f and consider differential symmetries of $f \circ \pi$.
- If $f \circ \pi$ has a differential symmetry, then the relation

$$Df(Ma, \pi x) + Df(\pi a, Mx) = \Lambda_M Df(\pi a, \pi x) \qquad (1)$$

holds for some M, where

$$Mx = \sum_{i=0}^{n-1} m_i x^{q^i} \text{ and } \Lambda_M x = \sum_{i=0}^{n-1} \lambda_i x^{q^i}.$$

Representation of $\mathbb{K}$

We use the following representation of $\mathbb{K}$:

$$\rho : \mathbb{K} \to \mathbb{A},$$

where $\mathbb{A} = \left\{ \begin{pmatrix} a & a^q & \dots & a^{q^{n-1}} \end{pmatrix}^\top : a \in \mathbb{K} \right\}$, defined by:

$$a \overset{\rho}{\mapsto} \begin{pmatrix} a & a^q & \dots & a^{q^{n-1}} \end{pmatrix}^\top =: \mathbf{a}.$$

Main Equation

Equation (1) can be expressed over $\mathbb{A}$ as follows:

$$\mathbf{a}^T(\mathbf{\Pi}^T\mathbf{DfM})\mathbf{x} + \mathbf{a}^T(\mathbf{M}^T\mathbf{Df\Pi})\mathbf{x} = \Lambda_M[\mathbf{a}^T(\mathbf{\Pi}^T\mathbf{Df\Pi})\mathbf{x}], \qquad (2)$$

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where:

- $\mathbf{Df}$ is the matrix representing Df as a bilinear form on $\mathbb{A}$ over $\mathbb{K}$,
- $\mathbf{M}$ is the matrix representation on $\mathbb{A}$ of the map $x \mapsto Mx$ and
- $\mathbf{\Pi}$ is the matix representation on $\mathbb{A}$ of the map $x \mapsto \Pi x$.

Df Matrix

$$f(x) = x^{q^\theta+1}$$

$$Df(a,x) = f(a+x) - f(a) - f(x) + f(0) = a^{q^\theta}x + ax^{q^\theta}$$

$$\mathbf{Df} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

M Matrix

$$Mx = \sum_{i=0}^{n-1} m_i x^{q^i}$$

$$\mathbf{M} = \begin{pmatrix} m_0 & m_1 & \cdots & m_{\theta-1} & m_\theta & m_{\theta+1} & \cdots & m_{n-1} \\ m_{n-1}^q & m_0^q & \cdots & m_{\theta-2}^q & m_{\theta-1}^q & m_\theta^q & \cdots & m_{n-2}^q \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n-\theta}^{q^\theta} & m_{n-\theta+1}^{q^\theta} & \cdots & m_{n-1}^{q^\theta} & m_0^{q^\theta} & m_1^{q^\theta} & \cdots & m_{n-\theta-1}^{q^\theta} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ m_1^{q^{n-1}} & m_2^{q^{n-1}} & \cdots & m_\theta^{q^{n-1}} & m_{\theta+1}^{q^{n-1}} & m_{\theta+2}^{q^{n-1}} & \cdots & m_0^{q^{n-1}} \end{pmatrix}$$

Π Matrix

$$\Pi x = \sum_{i=0}^{d} \beta_i x^{q^i}$$

$$\mathbf{\Pi} = \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_{d-1} & \beta_d & 0 & 0 & \cdots & 0 \\ 0 & \beta_0^q & \cdots & \beta_{d-2}^q & \beta_{d-1}^q & \beta_d^q & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_0^{q^d} & \beta_1^{q^d} & \beta_2^{q^d} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta_1^{q^{n-1}} & \beta_2^{q^{n-1}} & \cdots & \beta_d^{q^{n-1}} & 0 & 0 & 0 & \cdots & \beta_0^{q^{n-1}} \end{pmatrix}$$

Idea of Analysis

The following images assume $d < \theta + 1$. The argument in the paper is general.

$\Pi^\top D f M + M^\top D f \Pi$

If we shade the nonzero coordinates of the left hand side of (2), our matrix will look like the following:

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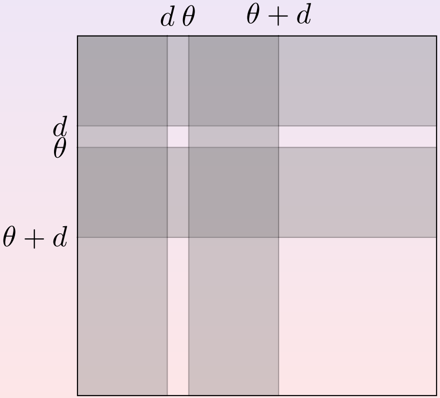

$\Lambda_M(\Pi^\top Df\Pi)$

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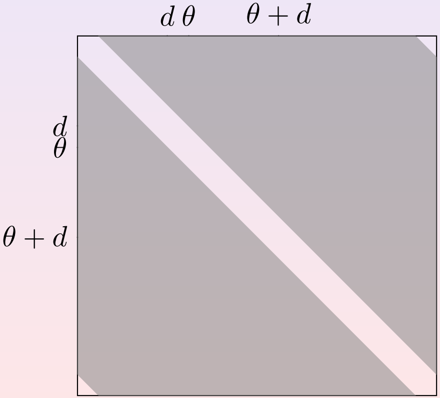
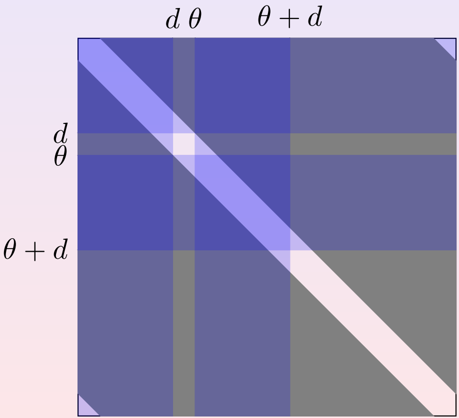


LHS=RHS, Equation (2)



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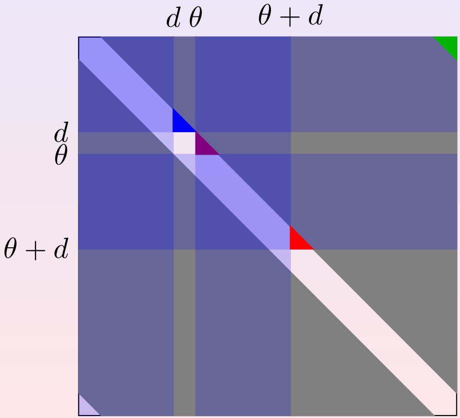
- Focus on coordinates for which (LHS=0) to determine for what r values is $m_r = 0$.
- Use this information to find for what values of r it is true that $\lambda_r = 0$ (RHS=0).
- Repeat until no new generic information is produced.

Single term LHS=0 (top half only, rest by symmetry)

Critical Regions

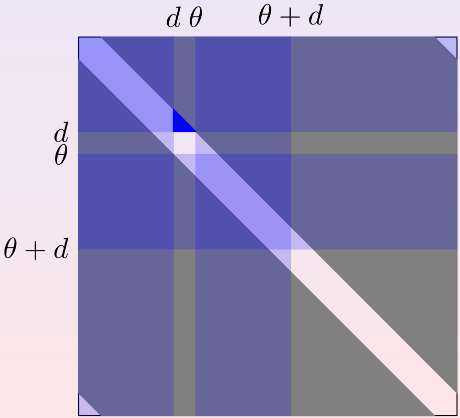


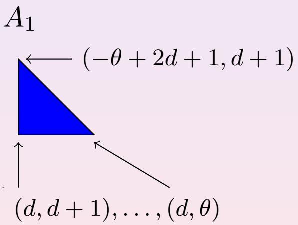
Let:

- Green= region A_2
- Blue= region A_1
- Purple= region B
- Red= region C

Coordinates of Regions

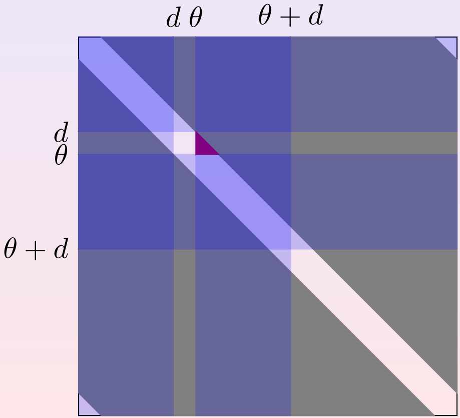
Region A_1

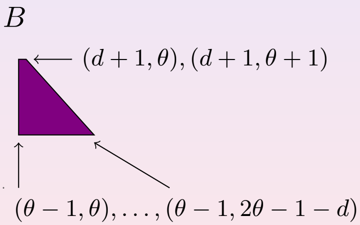




Coordinates of Regions

Region B





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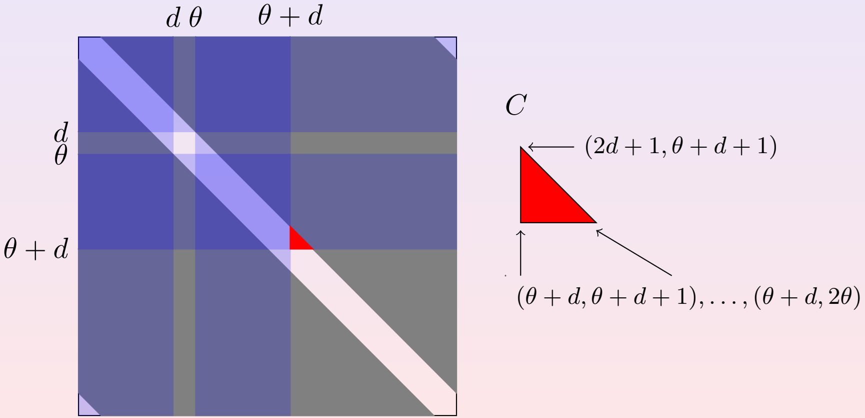
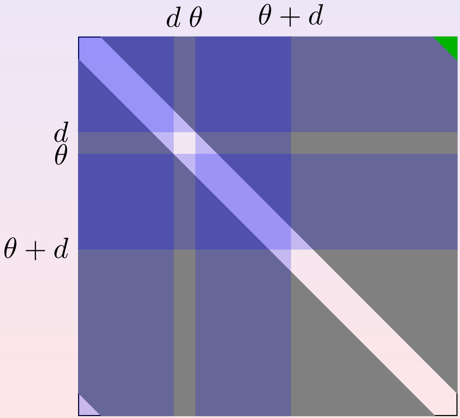
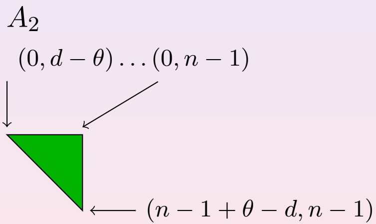
Region C



Coordinates of Regions

Region A_2





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Lemma

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Theorem

If Equation (2) *holds with the condition above and either* $d < \min\{\frac{n}{2} - \theta, |n - 3\theta|, \theta - 1\}$ *or* $d < \{\theta - \frac{n}{2}, |2n - 3\theta|, n - \theta - 1\}$, *then* $M = M_\sigma \circ \Pi$.

Conclusion on Symmetry

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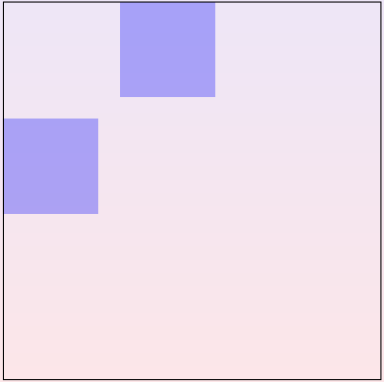
Key Choice for $d = 1$

$$\theta \in \left(2, \frac{n-1}{3}\right) \cup \left(\frac{n+1}{3}, \frac{n}{2} - 1\right) \cup \left(\frac{n}{2} + 1, \frac{2n-1}{3}\right) \cup \left(\frac{2n+1}{3}, n-2\right).$$

Q-Rank

We may consider PFLASH to have a central map of high degree but low Q-rank:

$$\mathbf{D}(f \circ \Pi) = \mathbf{\Pi}^\top \mathbf{Df}\mathbf{\Pi} =$$



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where $2 \leq \omega \leq 3$ is the linear algebra constant.

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For large r, as in all proposed parameters of PFLASH, this attack is no threat.

Parameters from [Chen et al., 2015]

Our analysis supports the security claims for the following parameters from [Chen et al.,2015]:

Scheme	Public Key (Bytes)	Security (bits)
PFLASH$(16, 62, 22, 1)$	39,040	80
PFLASH$(16, 74, 22, 1)$	72,124	104
PFLASH$(16, 94, 30, 1)$	142,848	128

Table: Security levels for standard parameters of PFLASH

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- Algebraically, PFLASH is a high degree HFE-.
- For realistic parameters, the Q-rank is sufficiently high to resist all variants of the KS-attack.

And as always, thanks for watching.

Thank you for your attention.
Questions?

References:

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