A Hybrid Lattice Reduction and Quantum Search Attack on LWE

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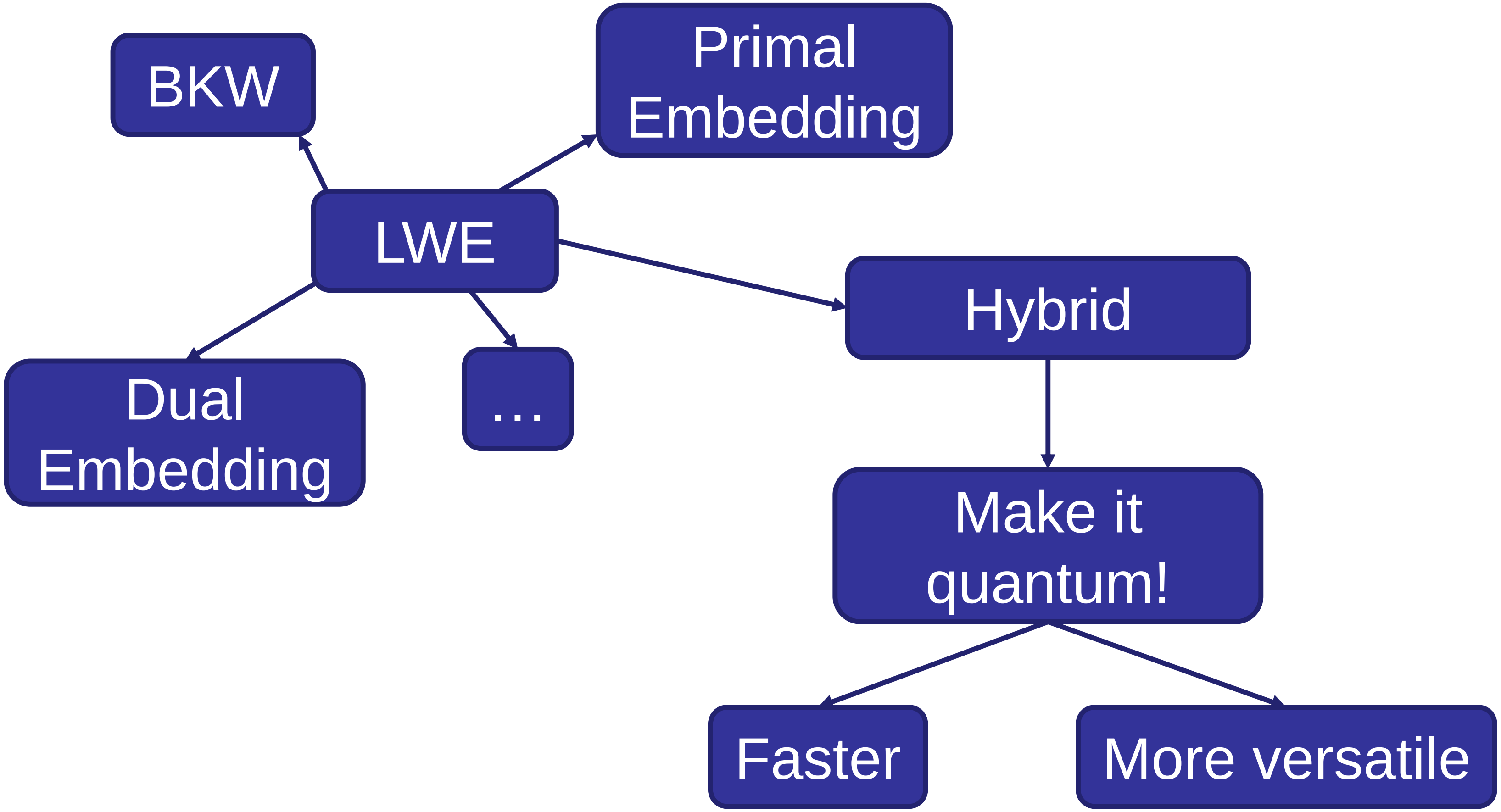
CROSSING

Motivation

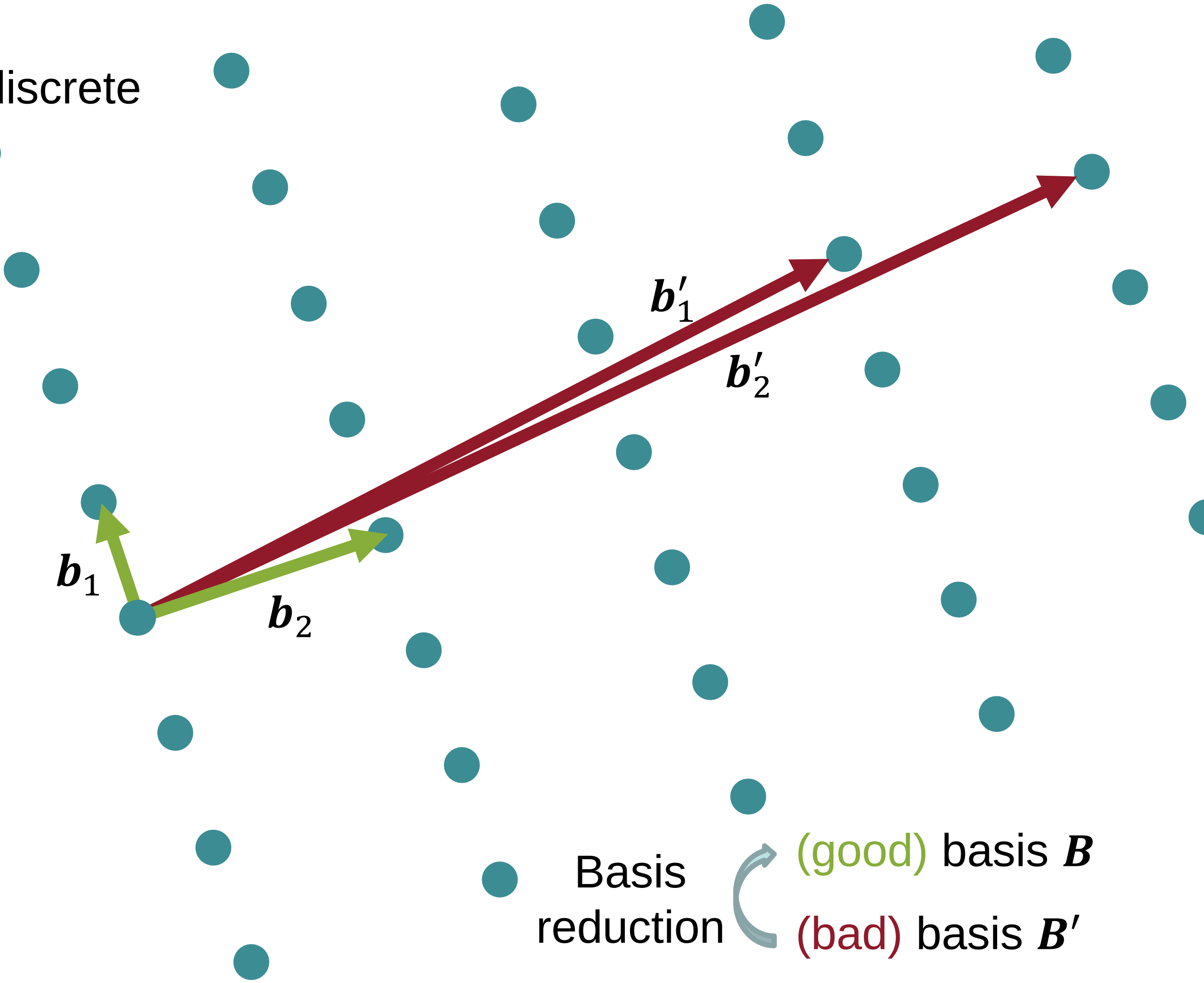

BKW
Primal Embedding
LWE
Hybrid
Dual Embedding
…
Make it quantum!
Faster
More versatile

Background and Notation

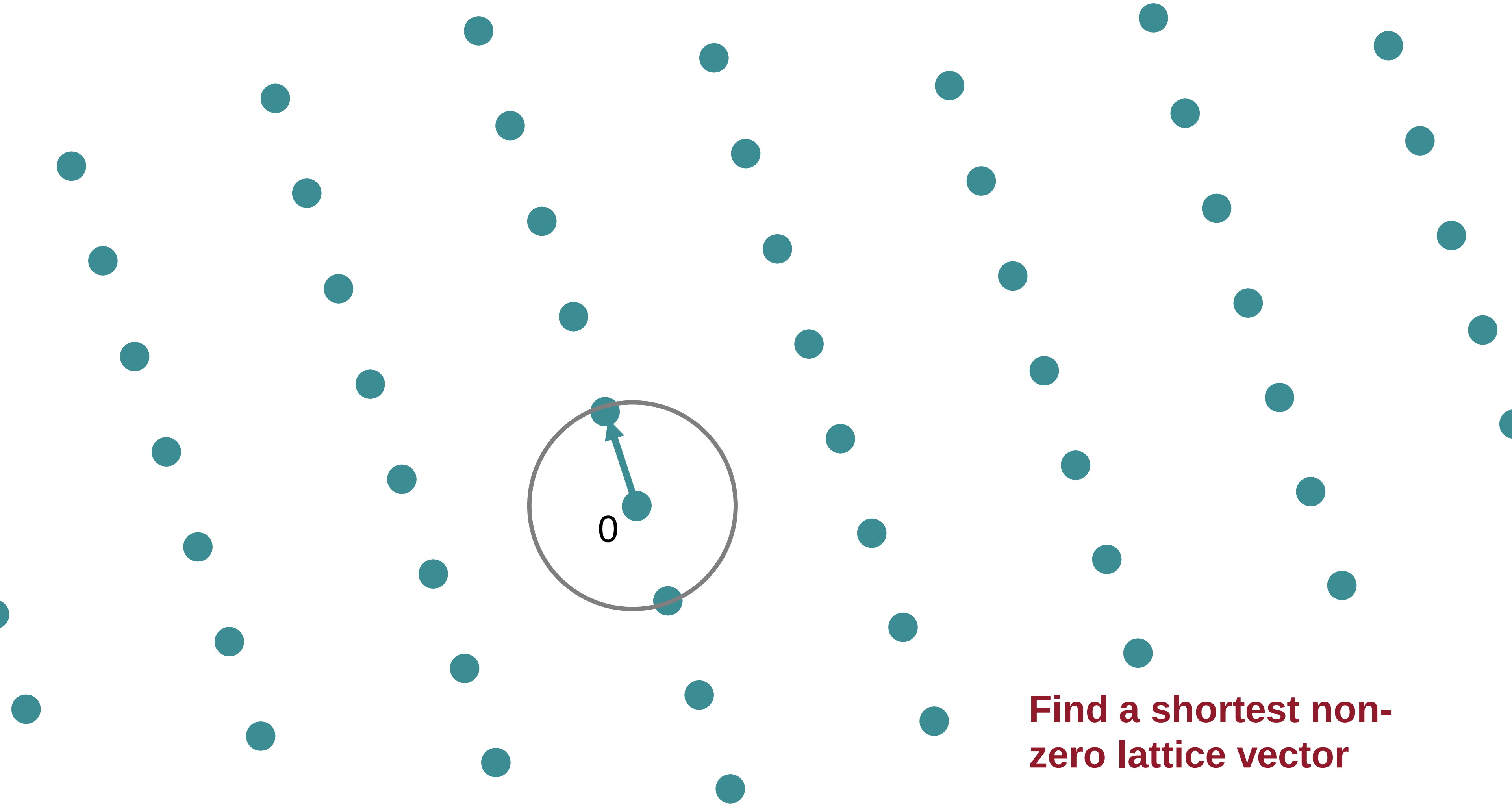
Lattices

n-dimensional **lattice** Λ: a discrete additive subgroup of $\mathbb{R}^n$

Basis of a lattice Λ: lin. ind. $\boldsymbol{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ such that $\Lambda = \mathbb{Z}\boldsymbol{b}_1 + \ldots + \mathbb{Z}\boldsymbol{b}_n$.

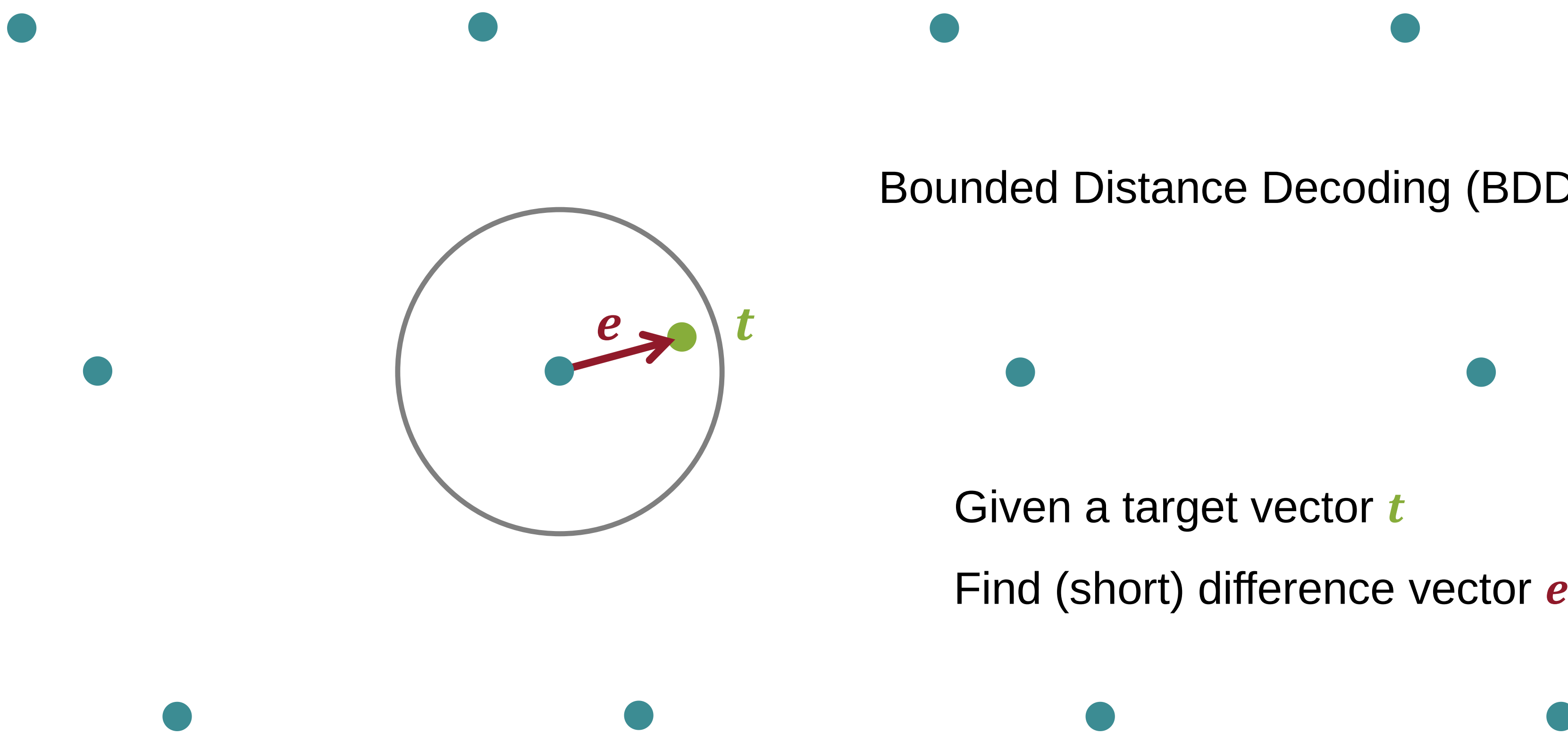


Shortest Vector Problem (SVP)



Find a shortest non-zero lattice vector

Closest Vector Problem (CVP)



Bounded Distance Decoding (BDD)

Given a target vector $\boldsymbol{t}$

Find (short) difference vector $\boldsymbol{e}$

Learning with Errors (LWE)

$$\boldsymbol{b} = \boldsymbol{A} \cdot \underbrace{\boldsymbol{s}}_{\text{short}} + \underbrace{\boldsymbol{e}}_{\text{short}} \mod q$$

Given: $\boldsymbol{A} \in \mathbb{Z}_q^{m\times n}$, $\boldsymbol{b} \in \mathbb{Z}_q^{m}$ Find: $\boldsymbol{s} \in \mathbb{Z}_q^{n}$

The (Quantum)
Hybrid Attack
on LWE

Our approach

We solve the LWE instance $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e} \bmod q$ as follows:

1. Transform LWE into SVP in some lattice Λ
2. Generate a basis $\boldsymbol{B}'$ of Λ of the form

$$\boldsymbol{B}' = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I}_r \end{pmatrix}$$

3. Solve SVP in Λ with our Quantum Hybrid Attack

Transforming LWE into SVP

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e} \; mod \; q$$

$$\Downarrow$$

$$\boldsymbol{v} = \underbrace{\begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{e} \\ 1 \end{pmatrix}}_{\text{short}} \in \Lambda = \{\boldsymbol{x} \in \mathbb{Z}^{n+m+1} : (\boldsymbol{A} | \boldsymbol{I}_m | -\boldsymbol{b})\boldsymbol{x} = \boldsymbol{0} \; mod \; q\}$$

The Quantum Hybrid Attack (Idea)

Setup: Find a shortest non-zero vector $\boldsymbol{v} \in \Lambda(\boldsymbol{B}') \subset \mathbb{Z}^d$, where $\boldsymbol{B}' = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I}_r \end{pmatrix}$

$$v = \begin{pmatrix} v_1 \\ \hline v_2 \end{pmatrix}$$

Find $\boldsymbol{v}_1 \in \mathbb{Z}^{d-r}$ with lattice-based techniques:

- Basis reduction as precomputation
- BDD-algorithms (Nearest Plane [Babai86])

Quantum-search for $\boldsymbol{v}_2 \in \mathbb{Z}^r$ (“Grover-like”)

Quantum vs. Classical Hybrid Attack

Quantum

Quantum search for $\boldsymbol{v}_2$

+ √-speed-up over brute-force

+ More versitile

+ Low memory consumption

+ No collision-finding probability

Classical

Meet-in-the-middle search for $\boldsymbol{v}_2$

+ √-speed-up over brute-force

- Requires highly structured keys

- Huge memory consumption

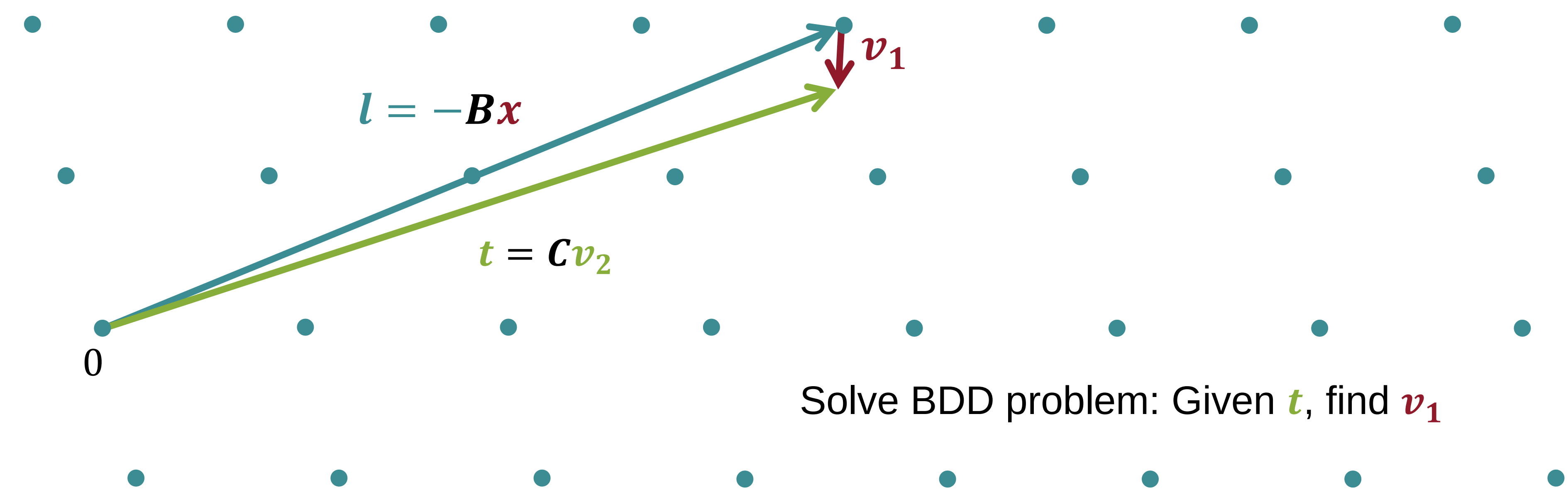
- Low collision-finding probability (might be $\approx 2^{-90}$)

The Attack

Find v_1 approach if v_2 is known

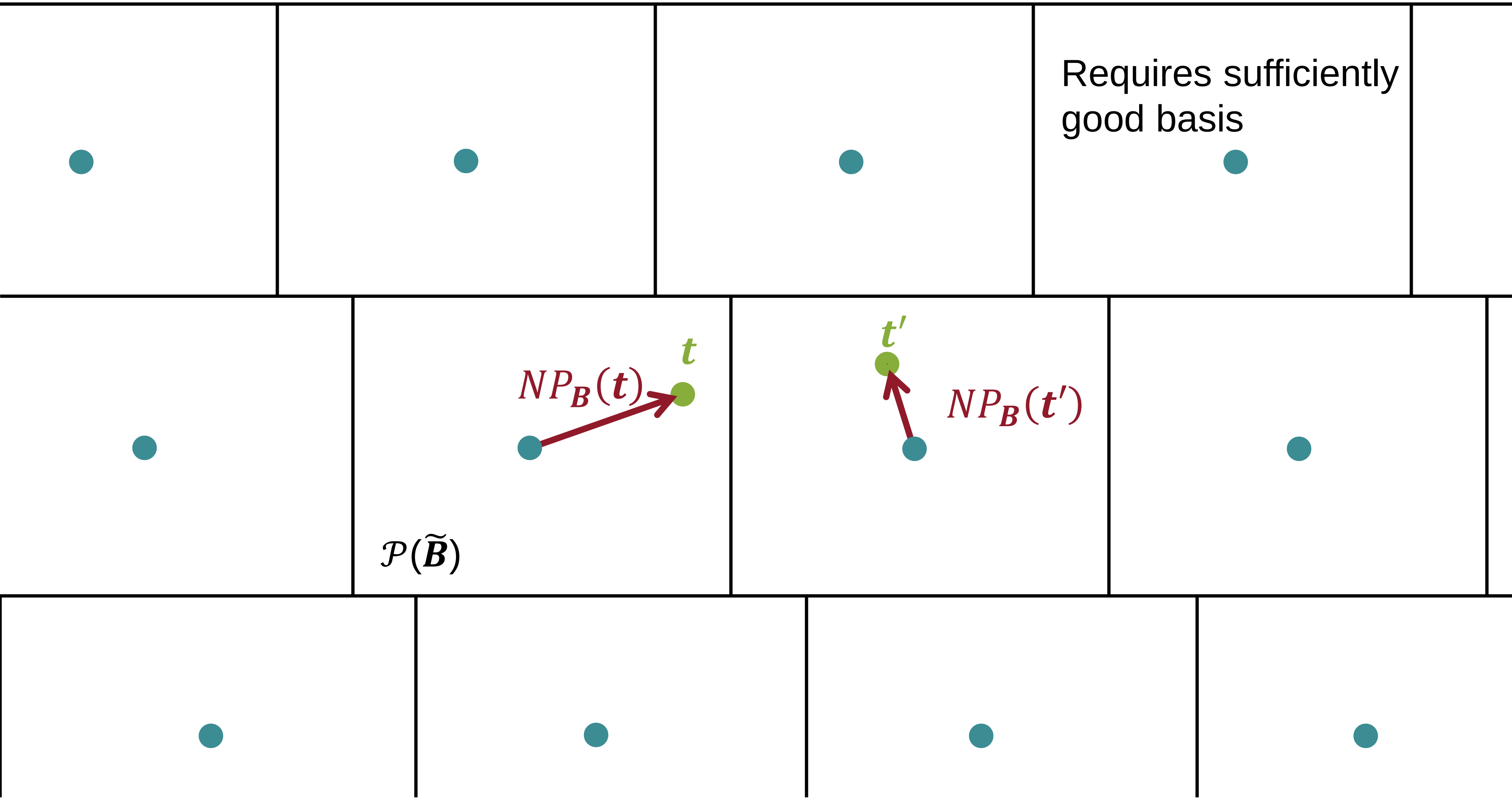
$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I_r} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{v_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Bx} + \boldsymbol{Cv_2} \\ \boldsymbol{v_2} \end{pmatrix}$$

Lattice $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{B})$



Solve BDD problem: Given $\boldsymbol{t}$, find $\boldsymbol{v_1}$

Solving BDD: Babai‘s Nearest Plane



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The Algorithm (Simplified Idea)

Task: find a shortest non-zero vector in a lattice Λ

Input: a search space $S \subset \mathbb{Z}^r$, a basis $\boldsymbol{B}' = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I}_r \end{pmatrix}$

<u>Loop</u>:
- "Quantum-guess" $\boldsymbol{v}_2' \in S$ (black box for now)
  - Check if guess is correct:
    - Calculate $\boldsymbol{v}_1' = NP_{\boldsymbol{B}}(\boldsymbol{C}\boldsymbol{v}_2')$
    - If $\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_1' \\ \boldsymbol{v}_2' \end{pmatrix}$ is sufficiently short
      - Return $\boldsymbol{v}$

Quantum Search (simplified)

- Let $S = \{s_1, \dots, s_k\}$ be a finite search space and $D = \{p_1, \dots, p_k\}$ be a probability distribution on S.
- Let $s \in S$ be a secret sampled from D. Task: find it!
- Choose a probability distribution $A = \{a_1, \dots, a_k\}$ on S.
- There exists a quantum algorithm (generalization of Grover's search algorithm) that finds s in roughly

$$L(A) = L(a_1, \dots, a_k) = \sum \frac{p_i}{\sqrt{a_i}}$$

loops (sampling from A and testing).

How to choose the distribution A

- Minimize the function $L(a_1, \dots, a_k) = \sum \frac{p_i}{\sqrt{a_i}}$ over all $a_1, \dots, a_k \in (0,1)$ with $a_1 + \cdots + a_k = 1$.
- Optimization with constraints in k variables (→ Lagrange)
- Optimal distribution $(\overline{a_1}, \dots, \overline{a_k})$ with $\overline{a_i} = \frac{p_i^{2/3}}{\sum p_i^{2/3}}$
- **Minimal number of loops**:

$$L_{min} = \left(\sum p_i^{2/3}\right)^{3/2}$$

Example (New Hope)

Take $S = \{-16, \ldots, 16\}^{200}$ and D to be the distribution on S given in the "New Hope" key exchange scheme [ADPS16]

- Classical brute-force search:

$$L_{classical} \approx 33^{200} \approx 2^{1009}$$

- Grover's quantum search:

$$L_{Grover} \approx \sqrt{33^{200}} \approx 2^{504}$$

- Our approach:

$$L_{our} \approx 2^{1.85 \cdot 200} \approx 2^{370}$$

Results

Runtime Analysis

Main result:

Let all notations be as before and $D = \{p_1, \ldots, p_k\}$ be the distribution from which v_2 is sampled.

Success probability:

$$p_{succ} \approx \prod_{i=1}^{m-r} \left(1 - \frac{2}{B\left(\frac{m-r-1}{2}, \frac{1}{2}\right)} \int_{-1}^{\max(-r_i,-1)} (1-y^2)^{\frac{m-r-3}{2}} dy \right)$$

where $B(\cdot,\cdot)$ denotes the Euler beta function, $r_i = \frac{R_i}{2\|\boldsymbol{v}_1\|}$ and R_i is the length of the i-th Gram-Schmidt vector in $\widetilde{\boldsymbol{B}}$.

Number of operations if successful: $\mathrm{T}_{hyb} \approx \frac{(m-r)^2}{2^{1.06}} \left(\sum p_i^{2/3} \right)^{3/2}$

Runtime Analysis

Remarks:

- T_{hyb} depends on the guessing-dimension r and the „quality“ δ (Hermite factor) of the basis $\boldsymbol{B}$
- Use precomputation (basis reduction) to change δ
- Balance precomputation and actual attack costs:

$$\mathrm{T}_{total}(r,\delta) = \frac{\mathrm{T}_{red}(r,\delta) + \mathrm{T}_{hyb}(r,\delta)}{p_{succ}(r,\delta)}$$

- Non-trivial optimization process in r and δ
- More details: see paper

Results

- Runtime depends on the cost of basis reduction (BKZ)
- How to model the SVP cost inside BKZ with block size β?
- Two (very) different ways in the literature:
  - Enumeration: $\mathrm{T}_{SVP} = 2^{0.27\beta\ln(\beta) - 1.019\beta + 16.1}$
  - Sieving: $\mathrm{T}_{SVP} = 2^{0.265\beta + 16.4}$
- $\mathrm{T}_{red} \approx dim * \#tours * \mathrm{T}_{SVP}$
- → We provide two different runtime estimates
- Compare our results with the LWE estimator (not claimed security levels!)

Results: New Hope and Frodo

Attack	New Hope	Frodo-592	Frodo-752	Frodo-864
Dual	1346	446	485	618
Decoding	833	-	-	-
Qu. Hybrid	**725**	**254**	**310**	**377**

Table 1: BKZ with enumeration

Attack	New Hope	Frodo-592	Frodo-752	Frodo-864
Dual	389	173	184	219
Decoding	380	-	-	-
Qu. Hybrid	**384**	**171**	**189**	**221**

Table 2: BKZ with sieving

Results: Lindner-Peikert

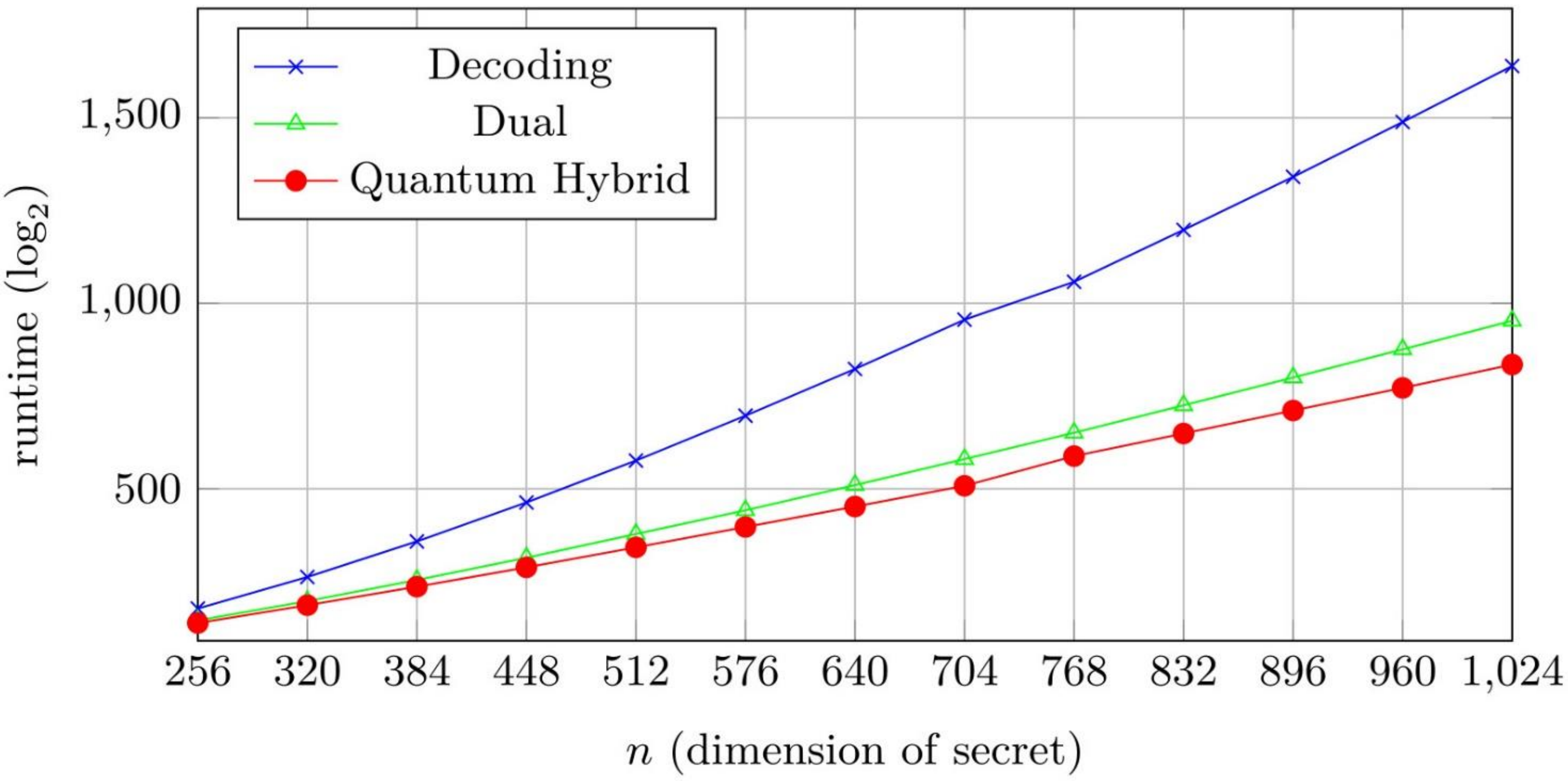

Figure 1: BKZ with enumeration

Results: Lindner-Peikert

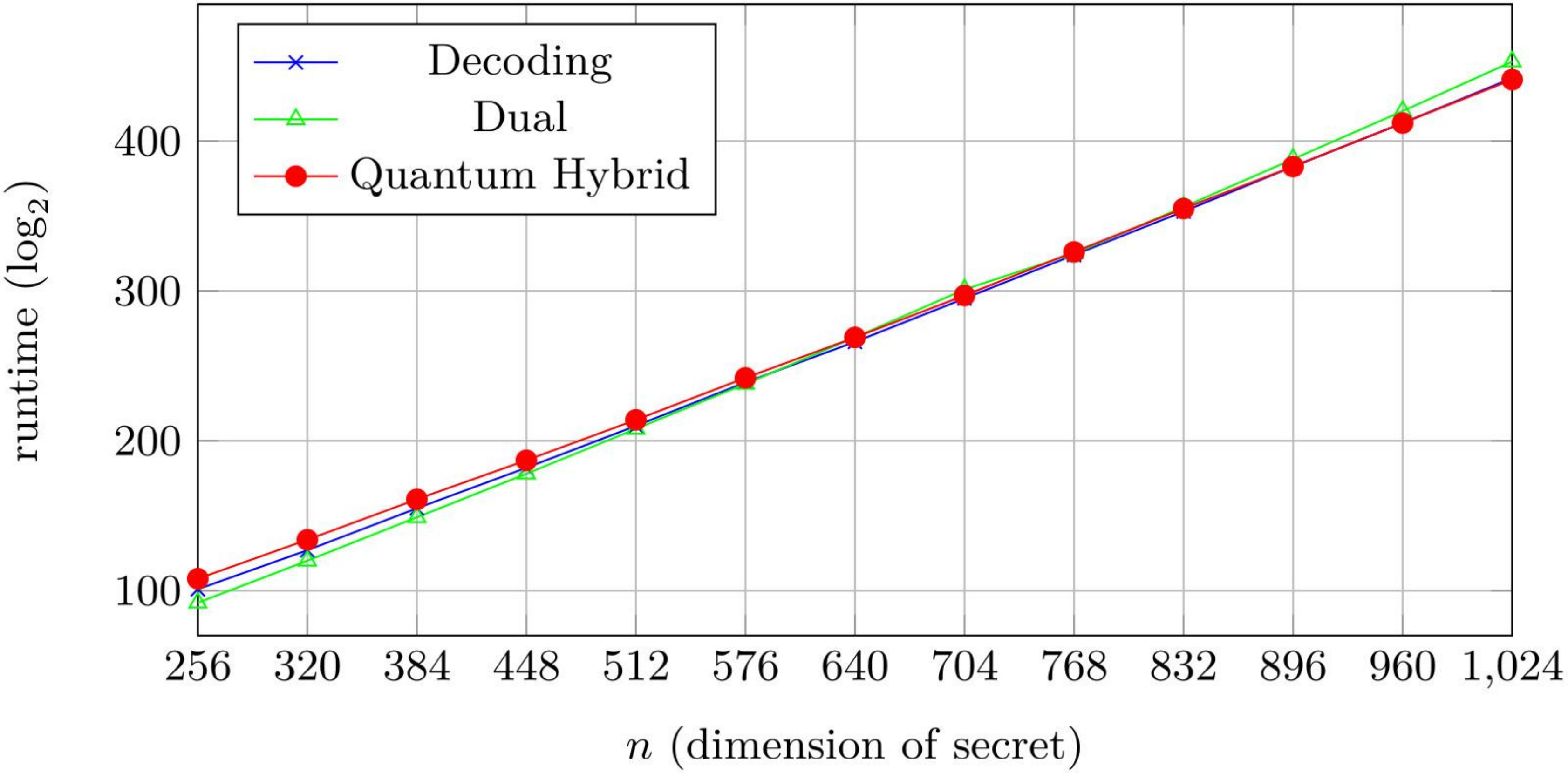

Figure 1: BKZ with enumeration

Conclusion

- New improved Quantum Hybrid Attack
- Detailed runtime analysis of the Quantum Hybrid
- New possibilities: apply Quantum Hybrid to non-uniform search spaces (e.g., LWE with Gaussian distribution)
- Outperforms other attacks in several instances

Thank you!

Questions?

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Literature

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